An arbitrage-free interpolation of volatilities

Nabil Kahalé describes a new construction of an implied volatilities surface from a discrete set of implied volatilities that is arbitrage-free and satisfies some smoothness conditions. His method provides an excellent fit to the smile of the local volatilities model, a standard extension of the Black-Scholes model known to be hard to calibrate in practice. This allows the pricing of exotic options in a way consistent with the smile.

It is well known that the implied volatilities of quoted European-style options are non-constant and depend both on the strike and maturity of the option, a phenomenon often referred to as the ‘smile’. Various models based on jump-diffusion, local or stochastic volatility have been proposed to explain and calibrate the smile (see Andersen & Andreasen, 2000, Derman & Kani, 1994, Dumas, Fleming & Whaley, 1998, Dupire, 1994, Heston, 1993, Lagnard & Osher, 1997, Li, 2001, Rebonato, 1999, and Rubinstein, 1994, and references therein.)

This article presents a new interpolation method for implied volatilities. If the market volatilities are arbitrage-free, we calculate an interpolating surface of the market volatilities for all strikes and maturities up to the last maturity that is arbitrage-free and satisfies some smoothness conditions. The basis block for our interpolation is a single-maturity interpolation with the following properties (the first holds only if there are no absolute discrete dividends):

1. If the input implied volatilities are constant, so are the interpolated volatilities.
2. The second derivative of the call price with respect to the strike is positive and continuous, as shown by numerical experiments.
3. The motivation behind the second property is that the second derivative of the call price is proportional to the implied density of the spot.

Our single-maturity interpolation does not depend on the shape of the discrete volatilities and applies to index, equity, forex and interest rate options. It takes seconds to calibrate a $10 \times 10$ volatilities matrix on a 800 Mhz processor and the quality of the fit is excellent. While our interpolation is of independent interest, one of its main applications is the calibration of the local volatilities model (Derman & Kani, 1994, Dupire, 1994, and Rubinstein, 1994), where the volatility of the spot is a deterministic function of the spot and time. The local volatilities can be calculated from the implied volatilities surface via Dupire’s formula (Dupire, 1994), which is very sensitive to the interpolation used. It is well known (Avellaneda et al., 1997) that, for standard interpolation methods, Dupire’s formula often leads to instabilities in the local volatilities. Our interpolated volatilities surface has been designed to calibrate Dupire’s model.

This allows the pricing of exotic options in a way consistent with the smile.

A one-dimensional interpolation, which numerical experiments show to be $C^2$. Then there is an overview of our two-dimensional interpolation algorithm and its application to the local volatilities model. In all the above, we assume we are in the equity market and there are no interest rates and no dividends. We then point out how dividends and interest rates are taken into account. An example on the S&P 500 index and a stability test are then given. Finally, we conclude. In Kahalé (2003), we give proofs that are omitted in this version for lack of space and show that our method can be easily extended to the forex market and, in the one-dimensional case, to the interest rate market.

Preliminaries

We use the following definition of arbitrage, which is slightly different from the one usually found in the literature. We define an arbitrage as a self-financing portfolio of securities that has a negative value today and a non-negative value at a given time in the future independently of the market behaviour. Thus one is certain to make profit by buying such a portfolio.

It can be shown that if the input implied volatilities are given for all maturities and strikes there is no arbitrage in the input if and only if the following conditions hold:

1. For a given maturity, the call price is non-increasing and convex with respect to the strike.
2. The call price is a non-decreasing function of time.

The convexity condition follows from the convexity of the payout at maturity. Based upon this result, one can find in linear time whether there exists an arbitrage within a discrete set of implied volatilities. In particular, the following can be shown in the one-dimensional case.

**Lemma 1.** Consider a sequence $(k_i, c_i)_{0 \leq i \leq n+1}$ such that:

$$0 = c_{n+1} = k_0 < k_1 < \cdots < k_n < k_{n+1} = \infty$$

and $c_i, 0 \leq i \leq n,$ is the price of a call with strike $k_i$. There is no arbitrage among these prices if and only if $c_n = \text{equal to the current spot, } c_0 \geq 0$ and:

$$-1 \leq \frac{c_i - c_{i-1}}{k_i - k_{i-1}} \leq \frac{c_{i+1} - c_i}{k_{i+1} - k_i} \leq 0 \text{ for } 1 \leq i < n$$

A one-dimensional $C^1$ interpolation method

We give in this section a $C^1$ arbitrage-free interpolation method for a given maturity. Like the cubic interpolation, our method is based on the concatenation of several functions. Moreover, these functions are convex. Our construction is inspired by the Black-Scholes formula. We start with the following lemma (Kahalé, 2003):

**Lemma 2.** Given $f > 0, \Sigma > 0, a$ and $b$, the function:

$$c(k) = c_{f, \Sigma, a, b}(k) = fN(d_1) - kN(d_2) + ak + b$$

where:

- $f$ is a function on $[a, b]$ with $f(a) = 0$ and $f(b) = 1$.
- $\Sigma$ is a positive constant.
- $N$ is the standard normal cumulative distribution function.
- $d_1$ and $d_2$ are given constants.
- $k$ is the strike.

The rest of the article is organised as follows. The first section contains preliminary results. We then give an arbitrage-free one-dimensional interpolation, which is provably $C^1$. Next we give an arbitrage-free one-dimen-
Theorem 1. For all real numbers $k_0, k_1, c_0, c_1$ and $c'_0$ such that $0 < k_0 < k_1$ and:

$$c'_0 < c_0 - c_0 < k_1 - k_0 < c_1 + 1 + c'_0$$

there exists a unique vector $(f, \Sigma, a, b)$ with $f > 0$, $\Sigma > 0$ such that the function $c = c_{\Sigma, a, b}$ satisfies the following conditions: $c(k_0) = c_0$, $c(k_1) = c_1$, $c'(k_0) = c'_0$ and $c'(k_1) = c'_1$. The vector $(f, \Sigma, a, b)$ is continuous with respect to $(k_0, k_1, c_0, c_1, c'_0, c'_1)$ and can be calculated numerically.

The proof is as follows. Given $a \in [c_0', 1 + c_1']$, let $d_2 = \log(f/k) + \Sigma^2/2$. Then:

$$d'_2 = \alpha \log(k) + \beta$$

for $i \in [0, 1]$. There exist $f > 0$ and $\Sigma > 0$ such that $\alpha = -\frac{1}{\Sigma}$ and $\beta = (\log(f)/\Sigma - \Sigma/2$. It follows that:

$$d'_2 = \frac{\log(f/k) - \Sigma^2/2}{\Sigma}$$

Note that $d'_2, \alpha, \beta, f$ and $\Sigma$ are continuous functions of $a$ as $a$ ranges in $[c_0', 1 + c_1']$.

Consider the function $c = c_{\Sigma, a, b}$ where $b$ is chosen so that $c(k_0) = c_0$. It follows from equation (6) that $c'(k) = \frac{d_2}{\Sigma} + a$, and so $c'(k) = c'_0$ by equation (5). It can be shown (Kahalé, 2003) that $c(k) = c_i$ for some $a \in [c_0', 1 + c_1']$.

Theorem 1 gives an interpolation method between two strikes. A similar method (Kahalé, 2003) can be used to extrapolate the call prices below or beyond a certain strike. The extrapolation satisfies limit conditions that are the same as in the constant volatility case. By combining the interpolated and extrapolated methods for a series of strikes, we obtain (Kahalé, 2003) the following.

Theorem 2. For all sequences $(k_i)_{i=0}^{s-1}, c_i)_{i=0}^{s-1}$ and $(c'_i)_{i=0}^{s-1}$ such that equation (1) holds together with the limit conditions:

$$c'_0 - c_0 < c'_{i+1} - c_{i+1} = 0 < c_i$$

and the convexity conditions:

$$c'_i < \frac{c_{i+1} - c_i}{k_{i+1} - k_i} < c'_i$$

for $0 \leq i < n$ for $i \in [0, 1]$. There exist a $C^1$ convex function $c(k), k > 0$ and a unique sequence $(f, \Sigma, a, b)_{i=0}^{s-1}$ such that $c(k) = c_{\Sigma, a, b}(k)$ on the interval $[k_i, k_{i+1}] = [0, \infty]$, $c(k) = c_i$ and $c'(k) = c'_i$ for $1 \leq i \leq n$, $i \neq j$, and the limit conditions defined in equations (9) and (10) hold. The sequence $(f, \Sigma, a, b)_{i=0}^{s-1}$ can be calculated numerically.

The proof is as follows. For $1 \leq i \leq n$, let $f_i = (c_i - c_{i-1})(k_i - k_{i-1})$. By theorem 2, for any $\gamma \in [f_i, f_{i+1}]$, there exist a $C^1$ convex function $c(k), k > 0$ and a sequence $(f, \Sigma, a, b)_{i=0}^{s-1}$ such that $c(k) = c_{\Sigma, a, b}(k)$ on the interval $[k_i, k_{i+1}] = [0, \infty]$, $c(k) = c_i$ for $1 \leq i \leq n$, $c'(k) = c'_i$ for $1 \leq i \leq n, i \neq j$, and $c$ has a continuous second derivative at $k_i$. Moreover, the limit properties in equations (9) and (10) hold. The sequence $(f, \Sigma, a, b)_{i=0}^{s-1}$ can be calculated numerically.

The algorithm description. We are now ready to describe our algorithm. Consider sequences $(k_i)_{i=0}^{s-1}$ and $(c_i)_{i=0}^{s-1}$ such that equation (1) holds, $c_0 > 0$ and:

$$-1 < \frac{c_i - c_{i-1}}{k_{i+1} - k_i} < \frac{c_{i+1} - c_i}{k_{i+1} - k_i} < 0$$

for $1 \leq i < n$.

Note that equation (11) is the same as equation (2) except that inequalities have been replaced by strict inequalities. Let $t > 0$ be an error parameter. Algorithm A consists of the following procedures:

1. Initialisation step. Let $c'_0 = -1, c'_i = 0$, and $c'_0 = (j_i + 1)/2$, for $1 \leq i \leq n$, where $j_i = (c'_i - c_{i-1})(k_{i+1} - k_i)$.

2. Loop. For $1 \leq j \leq n$, set $j_i = c(k)$, where $c$ is a function calculated in theorem 3 for the index $j$. Replace simultaneously $(c'(k))$ by $(j_i - y)$, $1 \leq j \leq n$. Repeat this step until $\max_{k \in [0, \infty]}|c'(k) - c'(k)| < \epsilon$.

Several methods can be used to improve the numerical stability and speed of convergence of algorithm A. We tested a variant of algorithm A, where the $c'_i$ and $y_i$ are updated in parallel using a Newton-Raphson method. Our experiments support our conjecture.

A two-dimensional interpolation method and Dupire’s model

If no arbitrage is found in the input implied volatilities, we calculate an arbitrage-free interpolating volatilities surface as follows:

1. We generate a one-dimensional arbitrage-free interpolation for each input maturity $t_i$ using the above variant of algorithm A.

2. For each maturity $t_i \in [t_{i-1}, t_{i+1}]$ and each strike $K$, we calculate the implied volatility $\sigma_{imp}(K, t_i)$ so that $\sigma_{imp}(K, t_i)^2 T$ is a linear interpolation of $\sigma_{imp}(K, t_i)^2 T$ for $K$.

3. We make the necessary adjustments so that the entire volatilities surface is arbitrage-free.

Our interpolation in the time domain ensures that if the no-arbitrage condition $C(K, t) < C(K, t_{i+1})$ holds, where $C(K, t)$ is today’s price of a European-style call with strike $K$ and maturity $T$, $C(K, t)$ is an increasing function of $T$ between $t_i$ and $t_{i+1}$. This is because, in the Black-Scholes formula, $C(K, t)$ depends on time only through $\sigma_{imp}(K, t_i)^2 T$. Moreover, $C(K, t)$ is an increasing function of $\sigma_{imp}(K, t_i)^2 T$. Thus the no-arbitrage condition with respect to time holds if and only if $\sigma_{imp}(K, t_i)^2 T$ is an increasing function of $T$.

In Dupire’s model (1994), the spot follows the following stochastic differential equation:

$$dS_t = \sigma_s S_t \, dW_t$$

Where $S_t$ is the spot price, $\sigma_s$ is the volatility, and $W_t$ is a standard Brownian motion.
Option pricing

Fifferential equation:
\[ dS_t = rS_t dt + \sigma(S_t, t) dW_t, \]
where \( W_t \) is a Brownian motion and \( \sigma(S_t, t) \) is a deterministic function. Dupire has shown that if the implied volatilities are known for all strikes and maturities then the local volatilities surface is uniquely determined.

More precisely:
\[ \sigma^2(K, T) = \frac{\frac{\partial c(K, T)}{\partial K}}{K^2 \frac{\partial^2 c(K, T)}{\partial K^2}}. \]

A classical problem in implementing the model is the instability of the local volatilities calculation. According to the practical cases we tested, our interpolating algorithm is well suited to calibrating Dupire’s model. This is because the second derivative with respect to the strike of the call prices generated by our algorithm exists in practice, and is continuous and positive for maturities up to the last input maturity. Moreover, the call prices derivative with respect to the maturity exists, and is continuous and positive except at input maturities. The local volatility surface is therefore, in general, continuous except at input maturities, and can be calculated approximately using finite difference approximations of derivatives.

It is known that the price of a contingent claim on \( S \) obeys the following partial differential equation:
\[ \frac{\partial u(S, t)}{\partial t} + \frac{1}{2} \sigma^2(S, t) S^2 \frac{\partial^2 u(S, t)}{\partial S^2} = 0, \]
where \( u(S, t) \) is the price at time \( t \) of a contingent claim if the spot price is \( S \) at time \( t \). The local volatilities surface can thus be used to calculate option prices using finite difference schemes such as the Crank-Nicholson’s algorithm.

It can be shown that, for a given maturity, the extrapolation methods above yield asymptotically constant implied volatilities as the strike goes to zero or to infinity. In practice, implied and local volatilities are set to a time-dependent constant for deep out-of-the-money and deep in-the-money strikes.

Dealing with dividends and interest rates. We assume that interest rates are deterministic and that absolute dividends exist up to a certain maturity, with no restrictions on proportional or continuous dividends. Let \( T^* \) be a maturity larger than all absolute dividends and relevant options maturities and \( S^* \) the forward of \( S \) at maturity \( T^* \). Thus \( S^* \) is continuous and drift-less and can be considered as an underlying in an interest rate and dividend-free world. We reduce the computation of the local volatilities surface to the case where interest rates and dividends are null using options on \( S^* \) and a transformation similar to the one in Overhaus et al (2002, section 4.6).

Example
Table A shows the implied volatilities matrix of the S&P 500 index in October 1995 given in Andersen & Brotherton-Ratcliffe (1998).

In figure 1, we plot the interpolated volatilities surface and in figure 2 the risk-neutral density function of the spot.

In table B, we compare the two-year Black-Scholes call prices using the interpolated implied volatilities and the input implied volatilities with the prices calculated using the local volatilities surface. The maximum difference is about 6 cents. Figures 3 and 4 show the relative errors (differences divided by the initial spot) between all input prices and the prices obtained via the Crank-Nicholson scheme or Monte Carlo simulation together with our local volatilities surface. We compared our method with other methods used in practice by replacing our single maturity interpolation method with a standard cubic spline interpolation method (Press et al, 1993) of implied volatilities. Implied volatilities are set constant beyond the smallest and largest input strikes, where the implied volatility derivative with respect to the strike is set to zero. The remaining steps of our algorithm are the same. Figure 5 shows the relative errors between input prices and the

<table>
<thead>
<tr>
<th>TxK</th>
<th>85%</th>
<th>90%</th>
<th>95%</th>
<th>100%</th>
<th>105%</th>
<th>110%</th>
<th>115%</th>
<th>120%</th>
<th>130%</th>
<th>140%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.175</td>
<td>0.190</td>
<td>0.168</td>
<td>0.133</td>
<td>0.113</td>
<td>0.102</td>
<td>0.097</td>
<td>0.120</td>
<td>0.142</td>
<td>0.169</td>
<td>0.200</td>
</tr>
<tr>
<td>0.425</td>
<td>0.177</td>
<td>0.155</td>
<td>0.138</td>
<td>0.125</td>
<td>0.109</td>
<td>0.103</td>
<td>0.100</td>
<td>0.114</td>
<td>0.130</td>
<td>0.150</td>
</tr>
<tr>
<td>0.695</td>
<td>0.172</td>
<td>0.157</td>
<td>0.144</td>
<td>0.133</td>
<td>0.118</td>
<td>0.104</td>
<td>0.100</td>
<td>0.101</td>
<td>0.108</td>
<td>0.124</td>
</tr>
<tr>
<td>0.940</td>
<td>0.171</td>
<td>0.159</td>
<td>0.149</td>
<td>0.137</td>
<td>0.127</td>
<td>0.113</td>
<td>0.106</td>
<td>0.103</td>
<td>0.100</td>
<td>0.110</td>
</tr>
<tr>
<td>1.000</td>
<td>0.171</td>
<td>0.159</td>
<td>0.150</td>
<td>0.138</td>
<td>0.128</td>
<td>0.115</td>
<td>0.107</td>
<td>0.103</td>
<td>0.099</td>
<td>0.108</td>
</tr>
<tr>
<td>1.500</td>
<td>0.169</td>
<td>0.160</td>
<td>0.151</td>
<td>0.142</td>
<td>0.133</td>
<td>0.124</td>
<td>0.119</td>
<td>0.113</td>
<td>0.107</td>
<td>0.102</td>
</tr>
<tr>
<td>2.000</td>
<td>0.169</td>
<td>0.161</td>
<td>0.153</td>
<td>0.145</td>
<td>0.137</td>
<td>0.130</td>
<td>0.126</td>
<td>0.119</td>
<td>0.115</td>
<td>0.111</td>
</tr>
<tr>
<td>3.000</td>
<td>0.168</td>
<td>0.161</td>
<td>0.155</td>
<td>0.149</td>
<td>0.143</td>
<td>0.137</td>
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<td>0.128</td>
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<tr>
<td>4.000</td>
<td>0.168</td>
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<tr>
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<td>0.159</td>
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<td>0.151</td>
<td>0.148</td>
<td>0.144</td>
<td>0.140</td>
<td>0.136</td>
<td>0.132</td>
</tr>
</tbody>
</table>

Note: the maturity \( T \) is expressed in years and the strike \( K \) as a percentage of the initial spot \( S_0 = $590 \). The interest rate is \( r = 6\% \) and the dividend yield is \( q = 2.62\% \).
### B. Two-year call Black-Scholes prices

<table>
<thead>
<tr>
<th>Strike in %</th>
<th>Input price</th>
<th>Interpolated price</th>
<th>Finite diff. price</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>125,7023</td>
<td>125,7023</td>
<td>125,6992</td>
</tr>
<tr>
<td>90</td>
<td>103,9506</td>
<td>103,9506</td>
<td>103,9517</td>
</tr>
<tr>
<td>95</td>
<td>83,5822</td>
<td>83,5821</td>
<td>83,5801</td>
</tr>
<tr>
<td>100</td>
<td>64,8987</td>
<td>64,8987</td>
<td>64,8965</td>
</tr>
<tr>
<td>105</td>
<td>48,2225</td>
<td>48,2226</td>
<td>48,2127</td>
</tr>
<tr>
<td>110</td>
<td>34,1869</td>
<td>34,1870</td>
<td>34,2147</td>
</tr>
<tr>
<td>115</td>
<td>23,6128</td>
<td>23,6124</td>
<td>23,5570</td>
</tr>
<tr>
<td>120</td>
<td>14,6757</td>
<td>14,6760</td>
<td>14,6960</td>
</tr>
<tr>
<td>130</td>
<td>5,6466</td>
<td>5,6466</td>
<td>5,6157</td>
</tr>
<tr>
<td>140</td>
<td>1,7778</td>
<td>1,7779</td>
<td>1,7569</td>
</tr>
</tbody>
</table>

Note: two-year call Black-Scholes prices using the input implied volatilities, the interpolated implied volatilities and the finite difference Crank-Nicholson prices using 100 time steps on the S&P 500 index in October 1995.

### C. Model 3 call prices

<table>
<thead>
<tr>
<th>Strike</th>
<th>Call price</th>
<th>Perturbed call price</th>
</tr>
</thead>
<tbody>
<tr>
<td>97</td>
<td>6,086</td>
<td>6,2395</td>
</tr>
<tr>
<td>98</td>
<td>5,3815</td>
<td>4,8847</td>
</tr>
<tr>
<td>99</td>
<td>4,7185</td>
<td>4,2616</td>
</tr>
<tr>
<td>100</td>
<td>4,0945</td>
<td>3,6782</td>
</tr>
<tr>
<td>101</td>
<td>3,5122</td>
<td>3,1367</td>
</tr>
<tr>
<td>102</td>
<td>2,9739</td>
<td>2,6391</td>
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<tr>
<td>103</td>
<td>2,4819</td>
<td>2,1871</td>
</tr>
<tr>
<td>104</td>
<td>2,0377</td>
<td></td>
</tr>
</tbody>
</table>

Note: call prices calculated using the model 3 local volatility original and perturbed function together with the Crank-Nicholson PDE method, with 100 time steps and 1,000 space steps, for $T = 0.25$.

### 3. PDE relative errors using our interpolation

Relative errors on call prices of the S&P 500 index in October 1995 using our local volatilities surface together with the Crank-Nicholson PDE method, with 100 time steps and 1,000 space steps.

### 4. MC relative errors using our interpolation

Relative errors on call prices of the S&P 500 index in October 1995 using our local volatilities surface together with the Monte Carlo method with 100 time steps and $10^5$ samples.

### 5. PDE relative errors using the cubic spline interpolation

Relative errors on call prices of the S&P 500 index in October 1995 using the cubic spline interpolation together with the Crank-Nicholson PDE method, with 100 time steps and 100 space steps.

### 6. Local volatilities: original versus perturbed

Model 3 local volatilities and local volatilities generated by our method for maturity $T = 0.25$ for the original and perturbed parameters.
prices obtained via the Crank-Nicholson scheme together with the cubic interpolation method. We set the number of space steps to 100 to diminish the relative errors. The maximum relative error using Monte Carlo simulation with the cubic interpolation method is about 9%. Monte Carlo simulation is very useful to price options involving several assets.

A stability test
We tested our method using the model 3 (Dumas, Fleming & Whaley, page 2,068) local volatility function:

\[ \sigma(K,T) = a_0 + a_1 K + a_2 K^2 + a_3 T + a_4 T^2 + a_5 K T \]

floored at 1% and capped at 400%, with \( a_0 = 105, a_1 = -1, a_2 = 0.001, a_3 = -0.2, a_4 = -0.0001, a_5 = 0.2, S_0 = 100, r = 0.05 \) and \( q = 0 \). We calculate the call prices for maturity \( T = 0.25 \) and strikes 97, 98, ... , 104 using the model 3 local volatility function together with the Crank-Nicholson partial differential equation method, with 100 time steps and 1,000 space steps. We then use our method to interpolate these call prices and generate a local volatility surface. The call prices calculated via the local volatilities generated by our method match the call prices calculated via the model 3 local volatilities. This is not surprising since there are infinitely many local volatility surfaces that generate the same input call prices for maturity \( T = 0.25 \) and strikes 97, 98, ... , 104.

Next, we multiply the parameters \( a_j \) of model 3 by 1.05 and repeat the same process as above. For the new parameters, the call prices calculated via the local volatilities generated by our method match the call prices calculated via the model 3 local volatilities with a relative error of order \( 10^{-5} \) for maturity \( T = 0.25 \) and strikes 97, 98, ... , 104. The local volatilities generated by our method do not match the model 3 local volatilities. This is not surprising since there are infinitely many local volatility surfaces that generate the same input call prices for maturity \( T = 0.25 \) and strikes 97, 98, ... , 104.

Conclusion
We have designed a one-dimensional interpolation algorithm for implied volatilities that is robust and has good smoothness properties. Our algorithm applies to equity, foreign and interest rate options. It can be extended to the two-dimensional case for equity and foreign options. In practice, the regularity properties of our interpolation scheme ensure it can be used to calibrate Dupire’s model.

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