SOPHIS

Robust rho hedging

Jean-Noël Dordain, Stéphane Mayrargue and Christophe Patry, members of the Sophis research team, present different methods to analyse the interest rate risk and how to design ‘optimal’i hedging portfolios.

For fixed-income financial assets, the rho exposure represents the market risk sensitivity, similar to the delta exposure for equity financial assets. However, if delta hedging is well understood and based on the calculation of a sensitivity to a single figure, the index spot price, the rho hedging strongly depends on the calculation of a sensitivity to the yield curve. Indeed, the short-term rho exposure is dramatically different from the long-term rho exposure. For instance, a three day rho moves one day ahead of 33% (1/3) although a one-year rho moves one day ahead of 0.27% (1/365). Thus, the hedging portfolio has to fit the rho of the hedged portfolio maturity by maturity.

The most common rho hedging financial assets are:

- ☐ interest rate futures (short-term risk), and
- ☐ bond futures or swaps (long-term risk).

Unfortunately, the rho term structure of these hedging instruments is highly non-trivial. One has to develop accurate methods to decompose the rho of the hedged portfolio in sensitivities to each hedging instruments.

Denote by $f(t)$ the term structure of rho exposure of the hedged portfolio, $(f_i(t))_i$ the term structure of rho exposure of the series $(I_i)_i$ of hedging instruments. We want to determine the quantities $(c_i)_i$ such that:

$$f(t) \approx \sum_i c_i f_i(t)$$

A naive solution would be to project $f(t)$ in the vector space spanned by the functions $(f_i(t))_i$ for a well-chosen scalar product. However, such an algorithm always fails as soon as $f(t)$ is sufficiently close to $\langle (f_i(t))_i \rangle^{\perp}$. For instance, if the hedged portfolio contains a single one-year zero coupon and if the hedging instrument is simply the two-year zero coupon, the hedging quantity calculated is equal to zero although one would expect the two-year zero coupon to fully hedge the one-year zero coupon.

We present two different approaches. If one limits the hedging instruments to interest rate futures, a direct calculation of the sensitivities of the portfolio to each interest rate futures is possible even if the inversion of the term structure of forward rates to zero-coupon rates is not a well-posed problem. The second approach is based on a constrained minimisation algorithm. In this case, the key issue is the determination of the time mesh for the calculation of $f(t)$, $(f_i(t))_i$ and the proper set of constraints to be verified.

I – Bumping the forward zero coupon rates

We directly calculate the sensitivity of the hedged portfolio to each interest rate future based on the same monetary rate (for example, EURIBOR 3M). To perform these calculations, we proceed as follows:

- ☐ The term structure of forward values of the EURIBOR 3M is denoted by $R(t)$. Given a time mesh $(t_i)_i$, we perform a bucket analysis bumping for each t_i the curve $R(t)$ to get a bumped EURIBOR 3M curve $R_i(t)$.
- ☐ For each i, the curve $R_i(t)$ is inverted to obtain the corresponding bumped zero-coupon curve $B_i(t)$.
- ☐ The profit and loss of the hedged portfolio is re-calculated using the bumped curve $B_i(t)$, which yields by finite difference the determination of:

$$\frac{\partial P\,\&\,L}{\partial EURIBOR3M(t_i)}$$

The first and third points do not present any difficulties. But, the computation of $B_i(t)$ from $R_i(t)$ is far from being trivial because the solution is not unique as one has to take into account banking days and holiday days.

To sketch the inversion algorithm, we use the following notations:

- ☐ T_0, the today date,
- ☐ $B_0(t,x)$, the initial zero coupon bond with an over-rate x,
- ☐ $B_i(t)$, the bumped zero coupon, and
- ☐ $2d$, the payment offset of the EURIBOR 3M, 3M being the maturity of the EURIBOR 3M.

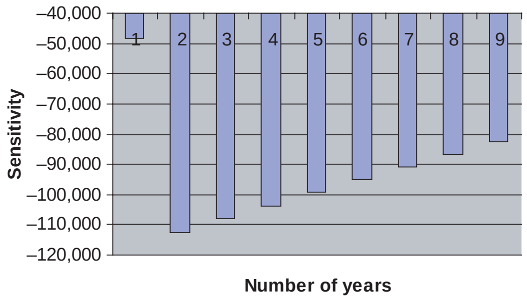
To take into account banking holidays, we introduce two operators:

- ☐ ⊕ add a time period taking into account banking holidays, and
- ☐ ∅ subtract a time period taking into account banking holidays.

The curve $B_i(t)$ is calculated inductively. $R_i(0)$ determines the value of: .

$$\frac{B_i(T_0 \oplus 2d \oplus 3M)}{B_i(T_0 \oplus 2d)}$$

Figure 1. Bond rho decomposition



The initial over-spread x_0 verifies:

$$\frac{B_i\left(T_0 \oplus 2d \oplus 3M\right)}{B_i\left(T_0 \oplus 2d\right)} = \frac{B_0\left(T_0 \oplus 2d \oplus 3M, x_0\right)}{B_0\left(T_0 \oplus 2d, x_0\right)}$$

For any t such that $T_0 \leq T \leq T_0 \oplus 2d \oplus 3M$, $B_i(t) = B_0(t, x_0)$.

We define x_j and y_j by induction: $x_0 = T_0 \oplus 2d$, $x_j = x_{j-1} \oplus 1$, $y_j = x_j \oplus 3M$.

We have two possibilities:

☐ $y_j > y_{j-1}$ (the point y_j is not the first point that corresponds to $R_i(x_j \varnothing 2d)$). $B_i(y_j)$ is calculated using the equation

$$\frac{B_i\left(y_j\right)}{B_i\left(x_j\right)} = f\left(R_i\left(x_j \varnothing 2d\right)\right)$$

where f is the forward discount factor function.

☐ $y_j = y_{j-1}$ (the point y_j is not the first point that corresponds to $R_i(x_j \varnothing 2d)$), the calculation of the zero coupon $B_i(y_j)$ is dropped.

Finally, j is set to $j + 1$ and the missing zero coupons are obtained by interpolation.

Numerical results

We consider a hedged portfolio with a single bond (10-year maturity, notional equal to 1 million, 5% nominal rate, yearly coupon payment frequency) on a 3% flat zero-coupon curve.

We obtain the following rho decomposition (see figure 1).

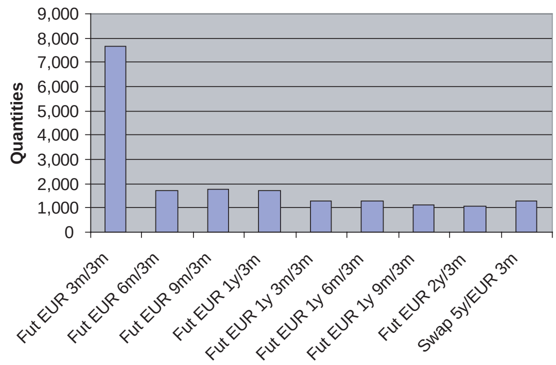
II – Constrained minimisation method

Let us denote by $(t_i)_i$ the time mesh used to perform the bucket analysis of the rho bumping directly the zero-coupon yield curve. Denote by n the number of hedging instruments and m the number of time meshes. We assume hereafter that $m \geq n$ (more bucket analysis than hedging instruments) to avoid an over-determined problem. Using the notations provided in the introduction, the minimisation coefficients $(c_j)_j$ are determined by the following minimisation criterion:

$$\underset{cj}{Min}\left[\sum_{i=1}^{m}\left(f\left(t_i\right) - \sum_{j=1}^{n} c_j f_j\left(t_i\right)\right)^2\right] \qquad (1)$$

under the constraint:

Figure 2. Hedging quantities



$$\sum_{i=1}^{m}\left(f\left(t_i\right) - \sum_{j=1}^{n} c_j f_j\left(t_i\right)\right) = 0 \qquad (2)$$

Let us write $f = F_1 + F_2$ where $F_1 \in \langle(f_j)_j\rangle$ and $F_2 \in \langle(f_j)_j\rangle^{\perp}$. The minimisation problem for F_1 only takes into account equation (1). The minimisation problem for F_2 only takes into account equation (2). Hence, in both cases, the global rho exposure is preserved and even for 'orthogonal' rho risk profiles (such as F_2), the algorithm provides a hedging portfolio, which perfectly hedges the global rho. As this algorithm is based on a projection, the $(c_j)_j$ obtained on the 'hedged portfolio – the hedging portfolio' are all equal to zero.

To practically solve the problem, one may perform the Legendre transform and get the min-max dual problem:

$$\underset{\lambda}{Min}\,\underset{cj}{Max}\left[\lambda\sum_{i=1}^{m}\left(f\left(t_i\right) - \sum_{j=1}^{n} c_j f_j\left(t_i\right)\right) - \sum_{i=1}^{m}\left(f\left(t_i\right) - \sum_{j=1}^{n} c_j f_j\left(t_i\right)\right)^2\right]$$

If $m = n$ $\langle(f_j)_j\rangle^{\perp} = \{0\}$, the constraint (2) is then fully included in the minimisation equation (1) and the solution may be found directly.

Numerical results

We consider a hedged portfolio with a single bond (five-year maturity, notional equal to 1 million, monthly coupon equal to 25,000) on a 5% flat zero-coupon curve. The hedging portfolio is composed as follows: eight futures on the EURIBOR 3M every month between three months and two years and an at-the-money interest rate swap paying a floating leg (written on the EURIBOR 3M, quarterly coupon payment frequency) and receiving a quarterly fixed leg (in linear and actual/360).

We obtain the following quantities $(c_j)_j$ for the hedging portfolio (see figure 2).

III – Conclusion

Sophis Risque System implements the two rho analysis methods and all the numerical figures were obtained using the system.

CONTACTS

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