One big happy **family**

One of the great challenges of the past two decades has been to keep up to date with the rapid evolution of option pricing models. In an update of an article first published in 1991, ***Charles Smithson*** *tracks the ongoing development of these models*

When I worked at one of the dealer banks in the early 1990s, I was assigned the task of reviewing the various option pricing models that were in use there and reporting on the appropriateness of the models to senior management. The reporting turned out to be the hard part: since 'trust me' was not going to do it, I needed to find a way to show a non-technical audience how I drew my conclusions about appropriateness. I realised that, since everyone understood Black-Scholes at one level or another, my best chance was to relate the other models to Black-Scholes. The easiest way I could think of to show the relation was a family tree.

The general equilibrium solution for the valuation of options provided in 1973 by Fischer Black, Myron Scholes and Robert Merton (hereafter the BSM model) was the end of the story in the sense that it provided a solution to a problem that economists had wrestled with for at least three-quarters of a century. The breakthrough was their recognition that (risky) shares and (risky) call options on those shares could be combined to construct a risk-free portfolio – so, the expected value of the payout of the option at maturity could be discounted using the risk-free rate.

However, the BSM model can also be considered the beginning of the story, in the sense that the three decades since its publication have seen a plethora of option pricing models. The family tree of option pricing models that appeared in *Risk* in 1991 (*Risk* October 1991, pages 37–44) was an attempt to organise the various models that had appeared to that date on the basis of the methods used for computing the option value (see box, Computational approaches).

While such a computational approach genealogy made sense in 1991, it no longer does today. Now, many of the models can be implemented using more than one computational approach. Consequently, in this encore, I will trace the genealogy of the option pricing models by first looking at the ways in which the original model was extended to treat additional payout profiles and at the ways the original approach was revised to fit market data.[1]

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Extensions of the BSM model

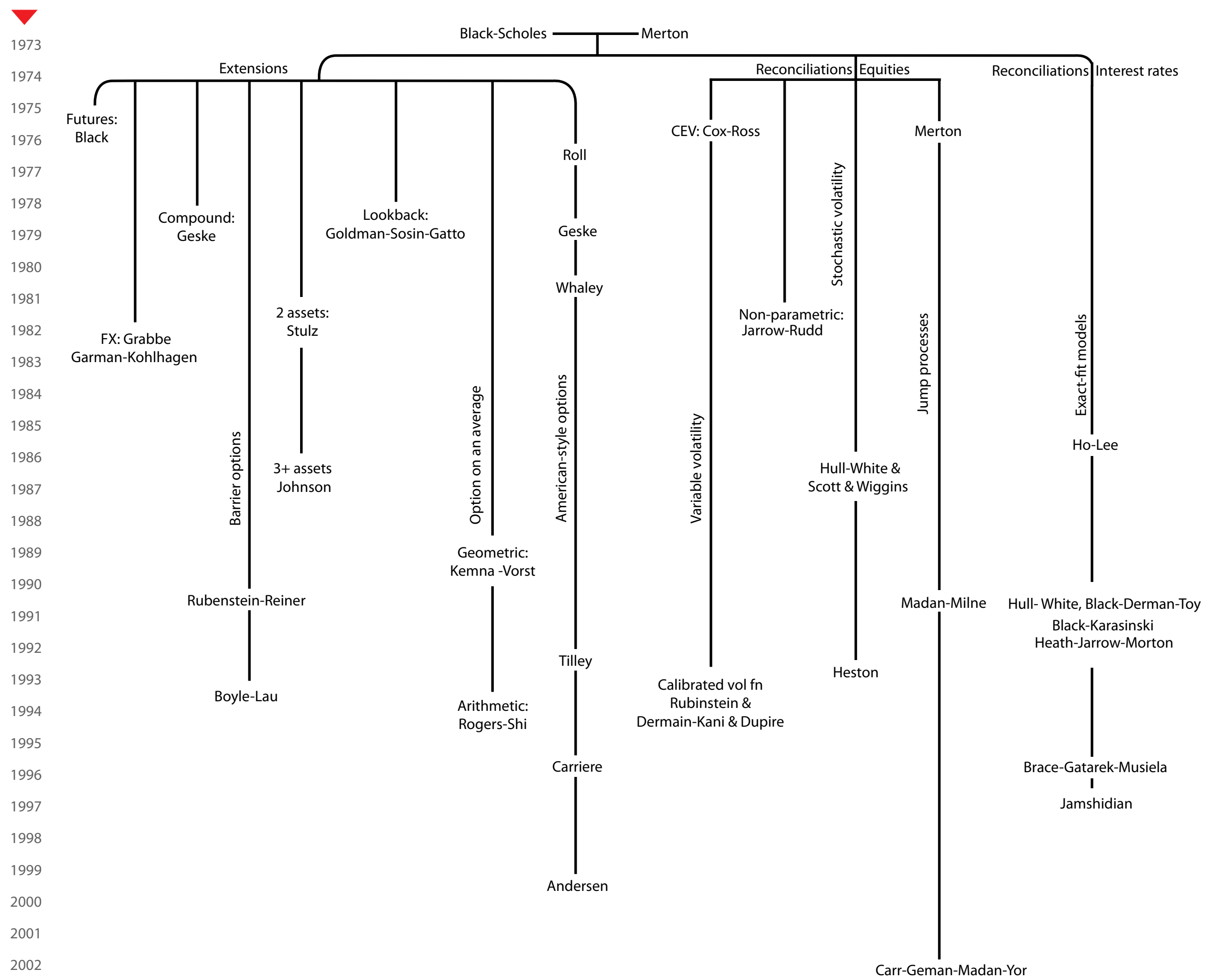
At its heart, valuing an option is an exercise in solving a partial differential equation[2] or obtaining the discounted expected value of the payout at maturity. While this exercise can be accomplished for any product and any payout function, there are some modelling contributions that deserve particular mention.

- **Options on futures.** In 1976, Fischer Black extended the BSM model to value options on futures contracts. Originally developed to price options on commodity futures, the Black model is now the market standard for valuing interest rate caplets and floorlets (using forward Libor as the underlying price and specifying volatility as the volatility of forward Libor) and for European-style swaptions (using the rate on a forward-start swap as the underlying price).
- **Options on currencies.** In 1983, Mark Garman and Steven Kohlhagen provided an analytic valuation model for European-style options on currencies using the approach Merton used for European options on dividend-paying stocks – that is, in the Garman-Kohlhagen model, the interest rate on the foreign currency takes the place of the dividend rate in the Merton continuous dividend model.

[1] *The reader who is interested in a more quantitative review is referred to the survey article by Mark Broadie and Jerome Detemple in the September 2004 issue of* Management Science

[2] *This is true for models that are path-independent – like the BSM model and other early option pricing models – or weakly path-dependent*

The options family tree



In the same year (indeed, in the same issue of the same journal), Orlin Grabbe provided an alternative specification for the value of European-style options on currencies using an approach similar to that used by Black for futures – he included the price of a domestic currency discount bond and the forward price of the foreign currency (determined by the variance of the forward foreign exchange rate).

The two approaches are mirror images of each other. Interest rate parity means that an option can be valued using either the spot rate and the foreign interest rate (Garman-Kohlhagen) or the forward rate (Grabbe).

Compound options. An analytic model for the valuation of compound options was provided by Robert Geske in 1979. Geske noted that the fundamental problem in using the Black-Scholes formula for the valuation of compound options is that Black-Scholes assumes the variance is constant, while in the case of options on shares, variance depends on the level of the share price (or, more fundamentally, on the value of the firm). When compared with the Black-Scholes model, the compound option model delivers higher values for deep-out-of-the-money options and near-maturity options, and lower values for deep-in-the-money options.

Path-dependent options.

Lookback options. In 1979, Barry Goldman, Howard Sosin and Mary Ann Gatto examined the pricing of a European-style call option where the exercise price is the minimum price of the share over the life of the option – an instrument now referred to as a lookback option. Subsequently, a number of option pricing models have been developed to consider not only lookback options but also other options for which the path of the underlying is important in valuation – for instance, options on the average price and barrier options (up-and-out, down-and-out, etc).

Options on an average. In 1990, AGZ Kemna and ACF Vorst provided an analytical solution for the value of an option on a geometric average. The more widely used option on the arithmetic average required a numerical model solution. In the lattice models, if the model is trying to keep track of option values corresponding to each value of the average at each node, the number of potential average prices increases exponentially with the number of time steps. In 1995, Chris Rogers and Z Shi published a model that bucketed a range of possible averages (rather than storing values for each possible average price) and then interpolated the price.

■ **Barrier options.** Mark Rubinstein and Eric Reiner published a closed-form valuation model for barrier options in 1991. Once a solution to the problem that the placement of the barrier relative to the discrete asset price levels has a large effect on accuracy in lattice and finite difference models was provided by Phelim Boyle and Sok Hoon Lau in 1994, numerical methods seem to have dominated the pricing of these options.

■ **American-style options.** An analytic valuation model for an American-style share option was developed by Richard Roll (1977), Robert Geske (1979) and Robert Whaley (1981). Binomial models have been widely used to value American-style options. Monte Carlo simulations would seem to be a natural method to use for valuing American-style options, but the difficulty is the need to simulate multiple paths for all possible stopping times before maturity.

> At its heart, valuing an option is an exercise in solving a partial differential equation or obtaining the discounted expected value of the payout at maturity

In 1993, Jim Tilley proposed simulation-based methods to price American-style options on a single underlying asset based on a bundling procedure – by partitioning the asset price space and grouping the paths that fall into a given partition at a fixed time as a bundle. More recently, researchers have focused on fast methods to compute approximations to the optimal exercise policy – for example, the regression-based approaches first proposed by Jacques Carriere in 1996 and the functional optimisation approach first proposed by Leif Andersen in 2000.

■ **Options on multiple assets.** Rene Stulz (1982) provided a two-asset model, while Herb Johnson (1987) extended the model to several assets.

Reconciliations of the BSM model with market data

Reconciling the option pricing models with market data presented difficulties for both equities and interest rates. However, the way in which modellers solved the difficulties differed dramatically. Modellers of options on equities and equity indexes focused on the random component – jumps and stochastic volatility. Modellers of options on bonds or interest rates focused on the drift term – calibrating the drift to match bond prices or developing a model in which this happens automatically.

■ **Options on equity and equity indexes.** In addition to assumptions about taxes, transaction costs[3] and interest rates, the BSM model was based on the assumption that share price behaviour could be modelled via a geometric Brownian motion process. This meant the price of the asset moves continuously (that is, no jumps) and that the volatility of the process is constant.

■ **Empirical inconsistencies.** There exist two types of empirical inconsistency between the BSM model and market observations. The first are inconsistencies between the data and the assumed geometric Brownian motion process. The geometric Brownian motion process results in the terminal (that is, at maturity) returns to the asset being distributed lognormally. Financial time series for almost any equity or equity index do not exhibit this normality for log-returns – the empirical distributions have fatter tails than does the normal distribution.

The others are inconsistencies between observed option prices and those implied by the BSM model. The implied volatility of an option is the volatility that equates the model price with the market price. The BSM model assumes that the volatility of the underlying asset is constant – that is, options of all strikes and maturities traded on the same underlying asset would have identical implied volatilities. In practice, option implied volatilities depend in a systematic way on the strike price and time to maturity – the option volatility smile or skew – and this dependence changes in an unpredictable way with the passage of calendar time.

While practitioners appear to have been content dealing with the volatility smile/skew via calibration, academics have proposed modifications to the Black-Scholes structure.

■ **Variable (albeit deterministic) volatility.** In 1976, John Cox and Stephen Ross proposed generalising the geometric Brownian motion process to a constant elasticity of variance process, so the diffusion coefficient would be a power-law function of the stock price. In 1994, Mark Rubinstein, Emanuel Derman and Iraj Kani, and Bruno Dupire all proposed methods to calibrate a volatility function of the stock price and time, such that the option pricing model matches actual option prices.

■ **Non-parametric modelling.** In 1982, Robert Jarrow and Andrew Rudd considered the case of prices following diffusion processes that do not necessarily generate lognormal distributions. In general, as the tails of the resulting distribution are fatter/thinner than those for a lognormal distribution, the Black-Scholes formula will undervalue/overvalue the option. They provided an adjustment to the Black-Scholes model that takes into account the discrepancies between the moments of the lognormal distribution and the true distribution.

■ **Stochastic volatility.** In 1987, John Hull and Alan White, Louis Scott and James Wiggins all proposed models that allowed volatility itself to be a stochastic process. It was, however, the stochastic volatility proposed by Stephen Heston in 1993 that has gained the most acceptance and use.

■ **Jump processes.** In 1976, Robert Merton combined the Brownian motion diffusion process with a Poisson jump process to provide a jump-diffusion process. After each jump, the price again follows the geometric Brownian motion process. (Subsequent models built on this approach by proposing compound Poisson jump processes).

In the early 1990s, researchers turned to the general class of jump processes called Lévy processes (which includes Brownian motion and Poisson as special cases). Lévy processes generate a wide class of jump-diffusion stochastic volatility models. Notable among the

[3] *In 1976, Jonathan Ingersoll expanded the BSM model to include a constant tax rate on dividends (*A theoretical and empirical investigation of the dual purpose funds, *Journal of Financial Economics 3(1), January–March) and in 1985, Hayne Leland expanded the model to permit transaction costs (*Option pricing and replication with transaction costs, *Journal of Finance 40(5), December)*

proposed models are the variance gamma model proposed by Dilip Madan and Franke Milne in 1991, and the model proposed by Peter Carr, Helyette Geman, Dilip Madan and Marc Yor in 2002.

Options on bonds/interest rates. Initially, practitioners valued options on bonds using the BSM model (adjusting for coupons in the same way that the equity option model adjusts for dividends). This worked reasonably well for short-term options, but did not work for longer-term options because it failed to capture pull to par – since a bond's value converges to par at maturity, the variance of the bond price distribution first increases with time and then decreases to zero at maturity.

There were attempts to deal with this problem via volatility specification, in the spirit of those used for equities.[4] However, by the mid-1980s, practitioners and academics realised that the option pricing model must include a model of interest rates that matches the observable term structure of bond prices. Such exact-fit interest rate models take the observed market yield curve as given and specify the stochastic evolution of the term structure so that there are no arbitrage possibilities.

Models based on the short rate. The first of the exact-fit interest rate option valuation models was introduced by Thomas Ho and Sang-Bin Lee in 1986. This one-factor model considers changes in the whole term structure rather than just changes in an interest rate. However, since it assumed that the short rate is normally distributed, negative interest rates are possible. And, while it permitted non-constant volatilities of interest rates, it required that all interest rates have the same volatility – all spot interest rates and all forward interest rates are equally variable.

In 1990 Fischer Black, Emanuel Derman and William Toy expanded on the Ho-Lee model by specifying a lognormally distributed short rate and a time-varying structure for volatility. This model incorporated mean reversion via a declining volatility curve.

Also in 1990, John Hull and Alan White proposed a model that incorporated not only a time-varying structure for volatility, but also another time-varying function that permits the future variability of short interest rates to be set independently of other data. More types of mean reversion could be incorporated. However, like the Ho-Lee model, the short rate was assumed to be normally distributed (so negative interest rates are possible).

The model proposed by Fischer Black and Piotr Karasinski in 1991 used a lognormal short rate and provided more flexibility in the mean-reversion process.

Models based on forward rates. In 1992, David Heath, Robert Jarrow and Andrew Morton (HJM) extended the exact-fit methodology. Their multi-factor model specifies all the instantaneous forward rates[5] in the future, such that the forward rates fit the current yield curve, and permits a wide range of volatility functions. It provides a flexible framework for building models that starts with the term structure of forward rates and assumptions about the evolution of forward rates. The more common implementations specified volatility functions that yielded either normally – or approximately lognormally – distributed changes in forward rates.

In 1996, Alan Brace, Dariusz Gatarek and Marek Musiela described a version of the HJM framework based on forward rates covering (realistic) finite intervals – for example, three-month periods, with lognormal distributions of changes in forward rates. This approach (often termed the Libor market model) rapidly gained widespread acceptance among derivatives dealers. An attractive feature is that this model is consistent with the market-standard Black model for pricing interest rate caplets and floorlets. A closely related model is the swap market model described by Farshid Jamshidian, in which the underlying is the rate of a forward-start swap.

The newest arrival

The first three decades since the appearance of the BSM model can be spilt into two regimes. The majority of the models that appeared in the first 15 years were extensions of the BSM model to underlyings other than equity and to payout structures other than the European-style option on the spot price. The majority of the models that appeared

Computational approaches

The Black-Scholes-Merton (BSM) model was an analytical model. In such models, a closed-form equation for the no-arbitrage value of the option is obtained by solving a partial differential equation (PDE).

The binomial (lattice) approach proposed by John Cox, Stephen Ross and Mark Rubinstein divides the time until option maturity into discrete intervals and presumes that, during each of the intervals, the price of the asset follows a binomial process – moving from its initial value, S, up to S_u (with probability p) or down to S_d (with probability $1 - p$). The option can be valued by moving down (that is, from maturity to origination) the resulting tree (or lattice) of asset prices. Beginning with one of the terminal asset values, the corresponding terminal value of the option can be calculated. Using the BSM insight that shares and calls can be combined to create a risk-free portfolio, the user can work back down the tree from time period T to time period T–1, discounting portfolio values in period T to period T–1 values using the risk-free interest rate. By continuing the process from T–1 to T–2 and so on, the value of the option at origination is obtained.

The finite-difference approach, initially suggested by Eduardo Schwartz and extended by Georges Courtadon, is based on finding a numerical solution to the PDE noted above – that is, the PDE is converted into a set of difference equations and the difference equations are solved iteratively. Michael Brennan and Eduardo Schwartz noted that the finite-difference approach is equivalent to a trinomial lattice approach, and John Hull and Alan White subsequently provided a modification that ensures that the trinomial lattice approach converges to the solution of the underlying differential equation.

Monte Carlo simulation, first proposed by Phelim Boyle, relies on the fact that the distribution of asset values at option expiry is determined by the process that generates future movements in the value of the asset. A simulation model can be viewed as progressing in three steps: 1) generate n random paths of the underlying variables; 2) compute the corresponding n discounted option payouts; and 3) average the results to estimate the expected value.

The analytic approximation approach combines the analytical and numerical techniques. For example, using this approach to value an American-style option, one would use one of the numerical techniques to estimate the premium for early exercise and then add this premium to the price of a European-style option obtained from an analytical model.

[4] *Clifford Ball and Walter Torous proposed a Brownian bridge process for the movement of the bond price in which the initial and the final bond prices are fixed (*Bond price dynamics and options, *Journal of Financial and Quantitative Analysis, December 1983) and Stephen Schaefer and Eduardo Schwartz proposed specifying the bond price volatility to be proportional to the duration of the bond (*A two-factor model of the term structure: an approximate solution, *Journal of Financial and Quantitative Analysis, December 1984)*

[5] *An instantaneous forward rate applies over an infinitesimally small future time interval. Actual forward rates over finite intervals (for example, three-month periods) are obtained as integrals of instantaneous forward rates*

in the second 15 years were modifications that reconciled the model to market data. The newest members of the option pricing family tree are models focused on credit. The late 1990s and early part of this decade witnessed the appearance of models to value default-risky bonds and loans. More recently, the focus of option modelling has been on credit derivatives and collateralised debt obligations. ■

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