# Asian basket spreads and other exotic averaging options

*Giuseppe Castellacci* and *Michael Siclari* of OpenLink introduce a class of exotic options that simultaneously generalises both Asian and basket options. They develop approximate analytic models for real-time pricing of complex instruments that average through time resets and different assets

Hedging and portfolio management requires increasingly flexible and efficient financial instruments. Asian and basket options[1] are a case in point, for such options' values depend on a time or weighted average of asset prices. They are therefore suitable for hedging exposure in a corresponding portfolio, such as a commodity or currency delivery occurring periodically during a specific time duration – an Asian call or put – or a basket of long or short assets – a basket call or put. One could hedge such positions with an equally weighted average of plain-vanilla options, although Asian and basket calls and puts provide a generally less expensive alternative.

Indeed, as many authors have observed, averaging has the effect of generally decreasing the variance of the state variable, thereby making the option less expensive. Moreover, the smaller the correlation between the different asset returns, the greater the chance that this decrease will occur. And mathematically, the parallelism between Asian and basket options is even more apparent than under the investment rationale. Both instruments' values depend on a weighted arithmetic average of asset prices. These averages, although not lognormal, can be modelled as such in several ways. The simplest models resort to approximating arithmetic averages with geometric averages[2], adjusting the strike for the discrepancy, as well as moment-matching arithmetic averages within the lognormal family[3].

In this paper we propose a theoretical framework for valuing exotic options on Asian- and basket-like averages, the most prominent example of which are Asian basket options. We introduce the concept of 'general arithmetic average', which simultaneously generalises Asian and basket-like averages. The gist of our approach to valuation consists of approximating such general averages with lognormal variables. Once this first reduction has been performed, one can apply known models for options whose payoff depends on lognormal variables.

After deriving approximate valuations for Asian basket options, we proceed to value Asian basket spread or general portfolio options. By doing so, we reduce the calculation to a spread option problem and come to the first example of a general averaging option that depends on more than one average – thereby crucially applying the formulas for the covariance of two general averages. For this valuation, we introduce a third class of models – hybrid models – that combine the features of both geometric average and lognormal valuation. We validate all the analytical approximation models computationally, by comparing them with full Monte Carlo simulations. Given its analytic nature, our computational approach is at least four orders of magnitude faster than Monte Carlo, and so essentially instantaneous – an obvious advantage for real-time trading.

## General averaging options

By general arithmetic average of a basket of $n$ assets with price processes $P_i(t)$ and fixings or resets $t_i (i=1, 2, \ldots, m)$, we mean the linear combination:

$$\mathbb{A} = \mathbb{A}(\mathbf{W},\mathbf{t}) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij} P_i(t_j) \, , \tag{1}$$

where the $w_{ij}$ are real weights. The corresponding general geometric average is defined as:

$$\mathbb{G} = \mathbb{G}(\mathbf{W},\mathbf{t}) = \prod_{i=1}^{n}\prod_{j=1}^{m} P_i(t_j)^{w_{ij}} \, . \tag{2}$$

In this framework, we can formulate a definition that includes Asian and basket options as well as their exotic counterparts. Note that the baskets and fixings implicit in each of the averages may, in general, be different. A general averaging (arithmetic) option is a derivative security whose payoff at expiration function $v$ depends on $p$ general arithmetic averages $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_p$, and a vector of parameters $\Phi$, such as strike price – that is:

$$v = v(\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_p, \Phi) \, . \tag{3}$$

Risk-neutral valuation gives as the arbitrage price of such options:

$$\begin{aligned} &V(\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_p, \Phi, t_0) = \\ &= e^{-r(T-t_0)} E^{RN} [v(\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_p, \Phi)] \end{aligned} \tag{4}$$

where $E^{RN}$ is the risk-neutral expectation, $r$ the short rate, $t_0$ the present time and $T$ time of expiration. For the three valuation approaches we are introducing, it is necessary to assume a dynamics for the prices. We assume that the price processes $\{P_i^{(s)}\} 1 \le i \le n(s)$ (for $s = 1, 2, \ldots, p$) jointly form an $N$-dimensional geometric Brownian motion process, where $N = \sum_{s=1}^{p} n(s)$ and $n(s)$ is the number of assets of the $s$th average. As a result, the joint volatility and correlation term structures are flat.

Table 1 lists the main types of general averaging options. We will present

Table 1: **Main types of general averaging options**

| Type | Average(s) | Payoff |
|---|---|---|
| Asian | $A = \sum_j w_j P(t_j)$ | max[±(A – K),0)] |
| Basket | $A = \sum_i w_i P_i$ | max[±(A – K),0)] |
| Asian basket | $A = \sum_{ij} w_{ij} P_i(t_j)$ | max[±(A – K),0)] |
| Asian spread | $A^{\pm} = \sum_j w_j^{\pm} P(t_j)$ | max[±(A$^+$ – A$^-$ – K),0)] |
| Basket spread | $A^{\pm} = \sum_i w_i^{\pm} P_i$ | max[±(A$^+$ – A$^-$ – K),0)] |
| Asian basket spread | $A^{\pm} = \sum_{ij} w_{ij}^{\pm} P_i(t_j)$ | max[±(A$^+$ – A$^-$ – K),0)] |

[1] *Basket options allow for the simultaneous management of exposures on a variety of underlying commodities such as natural gas, crude oil, coal, refined products and electricity. Asian options provide the same flexibility for commodity flows in time.*

[2] *See Vorst (1992) for Asian options, adapted to basket options by Gentle (1995)*

[3] *See Levy (1992) for Asian option and Milevsky & Posner (1998) for basket options)*

Figure 1: **Sampling of Monte Carlo model convergence**

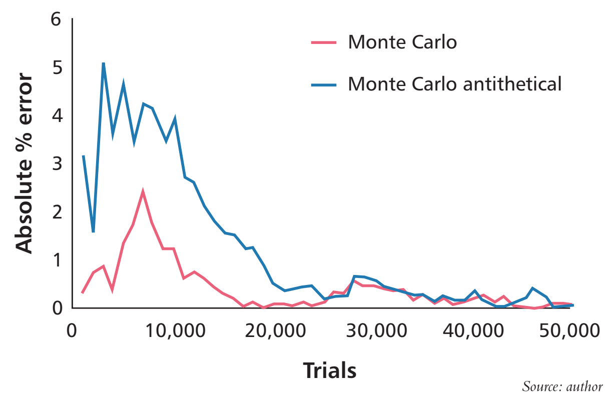


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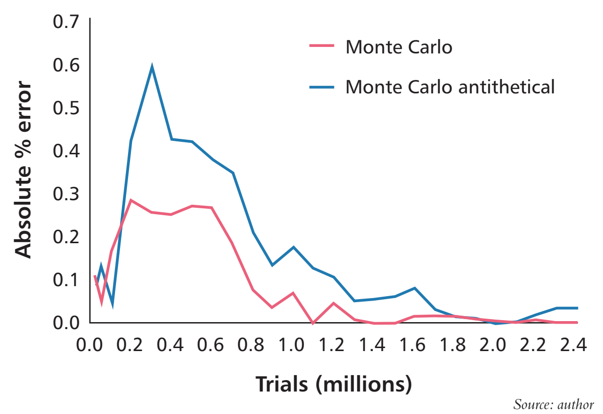
Figure 2: **Sampling of Monte Carlo model convergence**




*Source: author*

valuation models and freely refer to the terminology introduced here.

### *Generalising the Vorst model*

The idea of the Vorst model for pricing Asian options is to approximate an arithmetic average with a geometric one and then adjust the strike price by the difference of the expectations of the arithmetic and geometric average. Generalising this, we formulate the following principle:

- Given a payoff function $v=v(\mathbb{A}_1, \mathbb{A}_2,..., \mathbb{A}_p)$ depending on $p$ general averages, substitute the corresponding $\mathbb{G}_s$ for each $\mathbb{A}_s$, possibly adjust for the discrepancy between each $\mathbb{A}_s$ and $\mathbb{G}_s$, then use the corresponding valuation model applied to the adjusted payoff $v=v(\mathbb{G}_1,\mathbb{G}_2,...,\mathbb{G}_p)$.

Most multi-factor models require as input the first two moments of their factor returns. Given the lognormality of each $\mathbb{G}_s$, these can be easily calculated. For the mean:

$$E[\ln \mathbb{G}_s]=\sum_{i=1}^{n(s)}\sum_{j=1}^{m(s)} w_{ij}^{(s)} E[\ln P_i^{(s)}(t_j^{(s)})]= \sum_{i=1}^{n(s)}\sum_{j=1}^{m(s)} w_{ij}^{(s)}\left[\ln P_i^{(s)}(t_0^{(s)})+(r-q_i^{(s)}-(\sigma_i^{(s)})^2/2)(t_j^{(s)}-t_0)\right]. \tag{5}$$

The covariance between returns on averages $\mathbb{G}_q$ and $\mathbb{G}_s$ with weight matrices of size $n(q)\times m(q)$ and $n(s)\times m(s)$ respectively is:

$$\text{cov}(\ln \mathbb{G}_q, \ln \mathbb{G}_s)= =\sum_{i=1}^{n(q)}\sum_{k=1}^{n(s)} \rho_{ik}^{(qs)}\sigma_i^{(q)}\sigma_k^{(s)} \sum_{j=1}^{m(q)}\sum_{l=1}^{m(s)} w_{ij}^{(q)} w_{kl}^{(s)} \min[t_j^{(q)}-t_0, t_l^{(s)}-t_0], \tag{6}$$

where $\rho_{ik}^{(qs)}$ is the instantaneous correlation between $\ln P_i^{(q)}$ and $\ln P_k^{(s)}$ and $\sigma_i^{(q)}$ and $\sigma_k^{(s)}$ are the respective volatilities. In particular, the variance of the logarithm of one average is:

$$\text{Var}(\ln \mathbb{G})=\sum_{i=1}^{n}\sum_{k=1}^{n}\rho_{ik}\sigma_i\sigma_k\sum_{j=1}^{m}\sum_{l=1}^{m} w_{ij}w_{kl}\min[t_j-t_0, t_l-t_0]. \tag{7}$$

### *Generalising the lognormal model*

Levy (1992) proposed a model for Asian options that reduces the valuation problem to a Black-Scholes formula giving the discounted expectation of a payoff that depends on one lognormal variable. Levy's crucial insight was to produce such a lognormal variable from the Asian average by moment-matching the time average within the lognormal family of distributions. For basket options, the analog model has been traditionally known as 'lognormal'. These remarks suggest an extension to cover general averaging options according to the following principle:

- Given a payoff function $v=v(\mathbb{A}_1, \mathbb{A}_2,..., \mathbb{A}_p)$ depending on $p$ general averages, moment-match each $\mathbb{A}_s$ to a lognormal variable $X_s$ such that $[ln(X_1), ln(X_2),..., \ln(X_p)]\sim N_p(\mu,\Sigma)$. Then apply the corresponding valuation model to $v=v(X_1,X_2,...,X_p)$.

We need to calculate $\mu$ and $\Sigma$ or, equivalently, the expectations $\mu^{(s)}=E[ln(X_s)]=E[ln(\mathbb{A}_s)]$ and the covariances

$$\rho^{(qs)}\sigma^{(q)}\sigma^{(s)}=\text{cov}(\ln X_q, \ln X_s)=\text{cov}\left(\ln \mathbb{A}_q, \ln \mathbb{A}_s\right) \tag{8}$$

The expectations of the averages *per se* are easily calculated by linearity. Letting $F_i^{(s)}(t_j^{(s)})$ be the forward prices of the $i$th asset in the $s$th average with maturity at time $t_j^{(s)}$, we have

$$E[X_s]=E[\mathbb{A}_s]=\sum_{i=1}^{n(s)}\sum_{j=1}^{m(s)} w_{ij}^{(s)} F_i^{(s)}(t_j^{(s)}) \tag{9}$$

In most incarnations of this modelling approach, one assumes that the weights are renormalised so that $E[X_s]=1$. The expectation of the product of averages is:

$$E[X_qX_s]=E[\mathbb{A}_q\mathbb{A}_s]= =\sum_{i=1}^{n(q)}\sum_{j=1}^{m(q)}\sum_{k=1}^{n(s)}\sum_{l=1}^{m(s)} w_{ij}^{(q)}w_{kl}^{(s)}F_i^{(q)}(t_j^{(q)})F_k^{(s)}(t_l^{(s)})\times \times\exp\left(\rho_{ik}^{(qs)}\sigma_i^{(q)}\sigma_k^{(s)}\min[t_j^{(q)}-t_0, t_l^{(s)}-t_0]\right) \tag{10}$$

In order to determine the parameters $\mu^{(s)}$, $\sigma^{(s)}$ and $\rho^{(qs)}$ of the joint distribution of the $X_s$, we need to solve the equations.

$$E[X_s]=\exp\left(\mu^{(s)}+(\sigma^{(s)})^2/2\right) \qquad E[X_qX_s]=E[X_q]E[X_s]\exp\left(\rho^{(qs)}\sigma^{(q)}\sigma^{(s)}\right) \tag{11}$$

We can first solve for $\mu^{(s)}$ and $\sigma^{(s)}$ as:

$$\mu^{(s)}=2\ln E[X_s]-\tfrac{1}{2}\ln E[X_s^2] \qquad \sigma^{(s)}=\sqrt{\ln E[X_s^2]-2\ln E[X_s]}. \tag{12}$$

The correlations between the $q$th and the $s$th moment-matched averages – that is, the $\rho^{(qs)}$ can be obtained from the general formula

$$\rho^{(qs)}=\frac{1}{\sigma^{(q)}\sigma^{(s)}}\ln\left(\frac{E[X_qX_s]}{E[X_q]E[X_s]}\right). \tag{13}$$

Note that we obtain the actual values of the parameters by substituting the expectations obtained in equations 9 and 10.

## Asian basket options

In our framework, this is an option whose payoff at expiration time $v$ depends on one generalised average:

$$\mathbb{A}=\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}P_i(t_j) \tag{14}$$

where $n$ is the number of assets and $m$ the number of fixings, or resets. The payoff of an Asian basket option is $v=v(\mathbb{A},\Phi)$. We will value plain-vanilla Asian basket calls and puts, which have payoff $v=\max[\eta(\mathbb{A}-K),0]$), where $\eta=1$ for a call and –1 for a put.

Table 2: **Correlation matrix**

| | Item 1 | Item 2 | Item 3 | Item 4 | Item 5 |
|---|---|---|---|---|---|
| **Item 1** | 1.0 | 0.8 | 0.6 | 0.4 | 0.2 |
| **Item 2** | 0.8 | 1.0 | 0.8 | 0.6 | 0.4 |
| **Item 3** | 0.6 | 0.8 | 1.0 | 0.8 | 0.6 |
| **Item 4** | 0.4 | 0.6 | 0.8 | 1.0 | 0.8 |
| **Item 5** | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |

*Source: author*

Table 3: **Basket option parameters**

| | At-the-money strike | Disc factor | Time | No of items | Call/put |
|---|---|---|---|---|---|
| | 167 | 0.942539 | 0.98630 | 5 | 1 |
| | **Item1** | **Item 2** | **Item 3** | **Item 4** | **Item 5** |
| **Prices** | 50.00 | 35.00 | 38.00 | 19.00 | 25.00 |
| **Weights** | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| **Volatility** | 0.50 | 0.25 | 0.35 | 0.45 | 0.10 |

*Source: author*

### *Weight normalisation, past resets and put-call parity*

If the weights in $\mathbb{A}$ are not normalised to sum to unity, they can be normalised by using forward prices. Let $\mathbb{A}=\Sigma a_{ij}P_i(t_j)$ before any normalisation. Then we can (re-)normalise the weights defining:

$$w_{ij} = \frac{a_{ij}F_i(t_j)}{W} \tag{15}$$

where $F_i(t_j)$ is the forward price of asset $i$ at time $t_0$ with maturity $t_j$ and

$$W = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}F_i(t_j)$$

the forward price of $\mathbb{A}$. Note, that $W=1$ if the $a_{ij}$ are already normalised.

Another detail worth noting is that there are cases where some of the resets have already occurred. In this case, the option's payoff is decomposed into a deterministic and a stochastic part. The deterministic part contains past resets and is incorporated in the strike. The stochastic part is re-normalised to a new weighted average containing only future resets. The procedure is formally identical to the one for Asian options (see Levy (1992)). By the same token, it is easy to generalise Asian options' put-call parity to the case of Asian basket options.

### *Vorst-like valuation*

As in the standard Vorst model, the value of an Asian basket option depending on the underlying price $\mathbb{A}$ is approximated with the corresponding geometric average $\mathbb{G}$. To obtain explicit valuation formulas, the first two moments of $ln(\mathbb{G})$ are required. The moments are obtained from the general equations 5 and 6. Therefore, we have $E[\mathbb{G}]=e^{m_G+s_G^2/2}$, where $m_G=E[ln(\mathbb{G})]$ and $s_G^2=\text{var}(ln(\mathbb{G}))$. The strike price can be adjusted by subtracting the difference between the expectation of $\mathbb{G}$ and the expectation of $\mathbb{A}$ as follows

$$\tilde{K} = K/W - E[\mathbb{G}]+1 \tag{16}$$

For an Asian basket with normalised weights, the value of a European call or put is approximated by the formula:

$$V_\eta(\mathbb{A},K,t_0) = \eta e^{-r(T-t_0)}W\left[e^{m_G+s_G^2/2}N(\eta d_1) - \tilde{K}N(\eta d_2)\right]$$

$$d_1 = \frac{1}{s_G}\left[m_G + s_G^2 - \ln\tilde{K}\right] \qquad d_2 = d_1 - s_G. \tag{17}$$

### *Lognormal valuation*

Valuation according to the lognormal (Levy) model proceeds by computing the first two moments. Let $X$ be the lognormal variable that moment-matches $\mathbb{A}$. Then, the distribution of $X$ is completely determined by the first two moments of its logarithm, say $ln(X)\sim N(m_X,s_X^2)$. The general formulas 9 and 10 for the case of $p=1$ give the first two moments of $ln(X)$, as follows

$$m_X = -\tfrac{1}{2}\ln E[X^2] \qquad s_X^2 = \ln E[X^2]. \tag{18}$$

The value of the European call and put is approximated as

$$V_\eta(\mathbb{A},K,t_0) = \eta e^{-r(T-t_0)}\left[E[\mathbb{A}]N(\eta d_1) - KN(\eta d_2)\right] \tag{19}$$

where the notation is the same as above and

$$d_1 = \frac{1}{s_X}\left[-\ln K + s_X^2/2\right] \qquad d_2 = d_1 - s_X$$

## Asian basket spread options

In this section, we apply the previously developed framework for valuing general averaging options to the case of general portfolio options.

An Asian basket spread option or (general) portfolio option, is a derivative security whose payoff at expiration is:

$$v(P_1,P_2,...P_n;t_1,t_2,...,t_m;\Phi) = \max\left[\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}P_i(t_j) - K, 0\right], \tag{20}$$

where the $P_i$'s are prices of $n$ assets sampled at times $t_j$, such that $t_0<t_j\le T$ for all $j$'s, and the $a_{ij}$ are real constants. Such options are a generalisation of both Asian and basket spread options. Note that the coefficients $a_{ij}$ can have any sign. Hence, if $m=1$, the option can be thought of as a call on the difference of two basket prices, or a spread basket call. When $n=1$, the option can be thought of as a spread Asian call. It is customary to think of such cashflows as *Pay* (negative $a_{ij}$'s) and *Receive* (positive $a_{ij}$'s).

A natural way to model the value of such a portfolio is to separate its long (positive $a_{ij}$) and short positions (negative $a_{ij}$) to form two general averages:

$$\mathbb{A}_+ = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^+P_i(t_j) \quad \left(w_{ij}^+ = \frac{a_{ij}+|a_{ij}|}{2}\right)$$
$$\mathbb{A}_- = \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}^-P_i(t_j) \quad \left(w_{ij}^- = \frac{a_{ij}-|a_{ij}|}{2}\right). \tag{21}$$

In keeping with our overall modelling assumption that general averages are approximately lognormal, the payoff of a general portfolio option can be written as the payoff of a spread option on two general averages:

$$\max\left[\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}P_i(t_j) - K, 0\right] = \max\left[\mathbb{A}_+ - \mathbb{A}_- - K, 0\right]. \tag{22}$$

Any model for spread option valuation – for example, the Pearson semi-analytic model or the Wilcox model (see Pearson (1995)) – can be applied after approximating the $\mathbb{A}_\pm$ with suitable lognormal variables. The value

Figure 3: **Computational time v. error comparison**

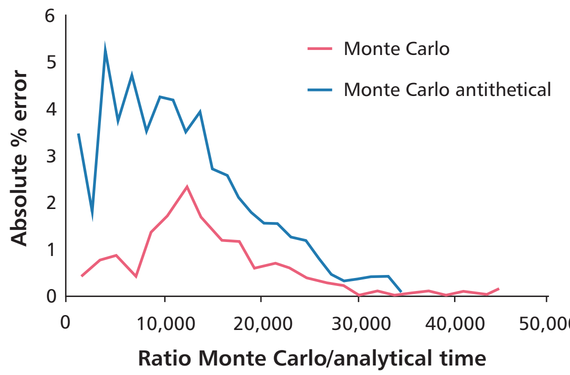



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Table 4: **Example 1 – option reset parameters**

| Reset date matrix | | | | |
|---|---|---|---|---|
| Time (days) | Time (days) | Time (days) | Time (days) | Time (days) |
| 1 | 7 | 14 | 21 | 30 |
| 31 | 37 | 44 | 51 | 60 |
| 61 | 67 | 74 | 81 | 90 |
| 91 | 97 | 104 | 111 | 120 |
| 121 | 127 | 134 | 141 | 150 |
| 151 | 157 | 164 | 171 | 180 |
| 181 | 187 | 194 | 201 | 210 |
| 211 | 217 | 224 | 231 | 240 |
| 241 | 247 | 254 | 261 | 270 |
| 271 | 277 | 284 | 291 | 300 |
| 301 | 307 | 314 | 321 | 330 |
| 331 | 337 | 344 | 351 | 360 |

*Source: author*

Table 5: **Example 2 – option reset parameters**

| Reset date matrix | | | | |
|---|---|---|---|---|
| Time (days) | Time (days) | Time (days) | Time (days) | Time (days) |
| 30 | 60 | 30 | 90 | 180 |
| 60 | 120 | 60 | 180 | 360 |
| 90 | 180 | 90 | 270 | |
| 120 | 240 | 120 | 360 | |
| 150 | 300 | 150 | | |
| 180 | 360 | 180 | | |
| 210 | | 210 | | |
| 240 | | 240 | | |
| 270 | | 270 | | |
| 300 | | 300 | | |
| 330 | | 330 | | |
| 360 | | 360 | | |

*Source: author*

of an Asian basket spread option is approximated with a model for the discounted expectation of

$$\max\left[X_+ - X_- - K, 0\right] \tag{23}$$

where the $X_\pm$ are jointly lognormal variables that approximate the $A_\pm$, respectively. All one needs to produce $X_\pm$ are formulas for the first two moments of the joint distribution of the $X_\pm$. These can be obtained following a generalised lognormal (with $p=2$) (see equations 9 and 10) or hybrid model, combining the former with Vorst's generalisation. Note that for this valuation, the correlation given by equation 13 is essential.

## Model validation

In order to validate the models, we will focus on the case of Asian basket and Asian basket spread options. We will also refer to each price process, or just the corresponding time average, as a 'basket item'. This convention and terminology correspond to common practice in the trading community.

### *Classical lognormal formulation for Asian basket options*

The variance of an Asian basket consistent with the lognormal model is given by the second equation in 18. Given the variance, we can easily derive a Black-Scholes formulation for the value of the option. This equation should apply to a basket option with an arbitrary number of items in the basket and an arbitrary number of fixings or reset dates per basket item. This model is used as a basis of comparison for two other models whose formulation is of a hybrid nature.

### *Hybrid models for Asian basket spread options*

Hybrid models are models for which several analytic approximation methodologies are combined or 'hybridised'. The characteristics of the option are broken down into several components, each of which is addressed individually with the appropriate analytical formulation. For the Asian basket model, we propose to calculate the adjusted volatility for each basket item according to the Vorst model for Asian options. Given the adjusted volatilities that account for the averaging, we treat the option as a regular basket option. We value the option using either the lognormal or Gentle (1995) treatment of non-averaging basket options. This results in two hybrid models for valuing Asian basket options, which we label Vorst-Lognormal and Vorst-Gentle.

Valuing Asian basket spread options introduces another level of complexity.To construct a model for this type of option, we extend all three of the above Asian basket spread option models by decomposing the basket items into the two basic components of the spread option and then applying a suitable spread option model for final valuation. Any spread option model will suffice. In this paper, we use the semi-analytical Pearson spread model as a good compromise between accuracy and speed.

Hence, the following examples test three analytical models to value generalised Asian basket options. They are the classical lognormal model and two hybrid models based on the Vorst model for time averaging and the Gentle and lognormal models for the baskets.

## Asian basket option model validation

In the following examples, we use five basket items to illustrate the application of the models to call options. The correlation matrix between the underlying assets of the basket is kept fixed and shown in table 2 (page 55). The various basket option parameters, such as prices and volatilities, are shown in table 3 (page 55). The weights in all examples are equal to one, which is not an oversimplification, as the initial prices are unequal. The underlying assets are assumed to be futures – hence, the expected futures prices are equal to the initial prices.

Tables 4 and 5 show the two sets of averaging or reset dates used in all the following examples.

Each set of resets might be tied to a commodity that is linked to a different expiration date sequence. Table 4 shows an extreme example, where the different commodities reset on the first day of the month, in the first, second and third weeks and on the last day of the month. The option is assumed to expire on the last reset date in the table. The prices are obtained approximately on a monthly basis for a year. Hence, this is a monthly averaged Asian basket option with five basket items.

The option examples used to validate the analytical models in the following sections are extreme cases in terms of the averaging characteristics and were designed to stress the models. The averaging dates are not necessarily realistic or representative of this class of options.

### *Monte Carlo Asian option models*

Before testing the analytic models against several examples, we discuss general Monte Carlo simulation techniques for two reasons. First, to estimate model accuracy, the values from the models are compared to full Monte Carlo Asian basket option simulations. Second, to give the motivation behind the introduction of the analytical models. Indeed, Monte Carlo valuation for path-dependent options, such as averaging options, has the drawback of requiring long computational times to be sufficiently accurate.

In particular, for the option given in example 1, all the 12 averaging dates per basket asset are unique. Price paths must be simulated with 60 time steps in order to generate correlated paths for the five basket items. Some thought was given to alternative Monte Carlo methods. We rejected quasi-random number sequences because they exhibit small, irremovable residual errors. We studied antithetic variance reduction to some extent. Figure 1 (page 54) shows the effect of antithetic variance reduction on an at-the-money option from example 1, where the strike is 167. The figure demonstrates the convergence for

Figure 4: **Model accuracy as a function of asset correlation – example 1**

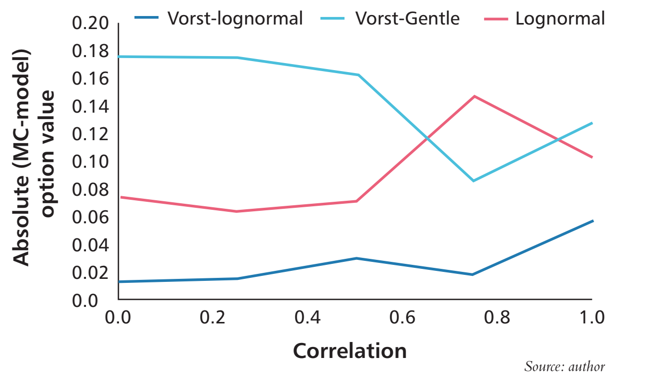


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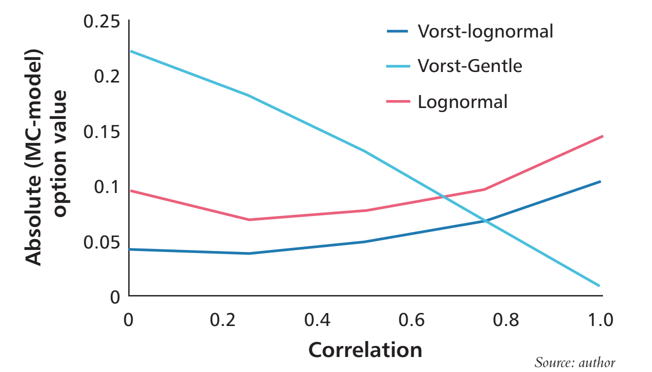
Figure 5: **Model accuracy as a function of asset correlation – example 2**

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the first 50,000 trials sampled every 1,000 trials. The error is a percentage error relative to a Monte Carlo value generated using 2.4 million trials.

The antithetic variance reduction has a noticeable impact in the first 25,000 trials in accelerating the convergence of the Monte Carlo simulation. Figure 2 (page 54) shows another comparison of convergence for the same Asian basket option, except that the simulation was carried out to 2.4 million trials sampled every 100,000 trials. A large sampling frequency will have a tendency to smooth out the convergence. Comparing the two figures shows that the percentage error can rise in further trials, even with the use of antithetic variance reduction. The error in the range of 200,000 to 600,000 trials is larger than the error observed in figure 1 at 50,000 trials. The antithetic variance reduction technique has the overall effect of reducing the errors by a factor of two. Trials in the order of 1 million need to be used in order to guarantee an absolute percentage error of less than 0.1% even with the use of antithetic variance reduction. Indeed, Monte Carlo simulations have been shown to converge with standard errors of the order of $Sdev/(Trials)^{1/2}$, where *Sdev* is the sample standard deviation of the option values during the simulation.

We chose 250,000 trials as our basic Monte Carlo simulation.. This number of trials gives us reasonable accuracy and resulted in computational times that were practical. 250,000 trials yielded a 0.037 standard error using the standard Monte Carlo valuation and a 0.021 standard error with antithetic variance-reduction techniques.

Figure 3 (page 55) addresses the issue of computational time of the analytical models relative to a Monte Carlo simulation. The figure illustrates the error versus the ratio of the Monte Carlo simulation to analytical computational time for the first 25,000 trials shown in figure 1 – again, sampled every 1,000 trials. If a 1% error tolerance level is acceptable and a 10,000-trial Monte Carlo simulation is used, the Monte Carlo simulation takes about 35,000 times more computational time than the analytic models.

Using the above criterion for accuracy in a Monte Carlo valuation, a single 10,000-trial option valuation on a 1.5 to 2.0 gigahertz PC, might only take three to five seconds. Why then is it important to have a very fast analytical model? Here we give some examples.

In general, a trader will need to compute sensitivities – Greeks – for this Asian basket option. To do this, three valuations of the option are required for each basket item per sensitivity. A full gamma matrix of sensitivities for our examples – that is, 5x5 – requires 45 valuations. Delta and vega require 10 valuations each. To calculate the above sensitivities, 60 option valuations are required. Our three to five seconds of computational time becomes three to five minutes per option.

What if a risk manager wants to compute value-at-risk (Var) using Monte Carlo simulation? At a minimum, 1000 trials would be required, yielding computational times of 1 to 2 hours for a single option, not to mention that the computational time for a small portfolio of these options would become impractical. These numbers demonstrate the need and the importance of having a fast and reasonably accurate approximate analytical model.

### *Example 1: equal number of fixings*

Table 4 (page 56) shows the 12 averaging or reset dates per basket item specified in days where today's date is zero. Table 6a (page 57) compares the analytical values for the three models to Monte Carlo values across a

Table 6a: **Comparison of option values for Asian basket option – example 1**

| Strike | Vorst-logN | Vorst-Gentle | LogN | Monte Carlo |
|---|---|---|---|---|
| 140 | 27.19 | 27.12 | 27.24 | 27.04 |
| 145 | 23.37 | 23.29 | 23.44 | 23.22 |
| 150 | 19.85 | 19.75 | 19.93 | 19.71 |
| 155 | 16.64 | 16.54 | 16.74 | 16.54 |
| 160 | 13.78 | 13.67 | 13.88 | 13.73 |
| 165 | 11.27 | 11.15 | 11.38 | 11.28 |
| 167 | 10.36 | 10.24 | 10.47 | 10.40 |
| 170 | 9.103 | 8.983 | 9.210 | 9.178 |
| 175 | 7.263 | 7.147 | 7.368 | 7.400 |
| 180 | 5.727 | 5.617 | 5.826 | 5.917 |
| 185 | 4.464 | 4.363 | 4.556 | 4.693 |
| 190 | 3.441 | 3.351 | 3.525 | 3.700 |

*Source: author*

Table 6b: **Comparison of absolute errors for Asian basket option – example 2**

| Strike | Vorst-logN | Vorst-Gentle | LogN |
|---|---|---|---|
| 140 | 0.14 | 0.08 | 0.20 |
| 145 | 0.15 | 0.07 | 0.22 |
| 150 | 0.14 | 0.04 | 0.22 |
| 155 | 0.10 | 0.00 | 0.20 |
| 160 | 0.05 | 0.06 | 0.15 |
| 165 | 0.01 | 0.13 | 0.10 |
| 167 | 0.04 | 0.16 | 0.07 |
| 170 | 0.08 | 0.19 | 0.03 |
| 175 | 0.14 | 0.25 | 0.03 |
| 180 | 0.19 | 0.30 | 0.09 |
| 185 | 0.23 | 0.33 | 0.14 |
| 190 | 0.26 | 0.35 | 0.17 |
| Mean | 0.13 | 0.16 | 0.14 |
| Standard dev. | 0.08 | 0.12 | 0.07 |

*Source: author*

Table 7a: **Example 2 – comparison of option values for Asian basket option**

| Strike | Vorst-logN | Vorst-Gentle | Lognormal | Monte Carlo |
|---|---|---|---|---|
| 140 | 27.95 | 27.84 | 27.97 | 27.76 |
| 145 | 24.30 | 24.18 | 24.32 | 24.11 |
| 150 | 20.93 | 20.79 | 20.96 | 20.75 |
| 155 | 17.86 | 17.70 | 17.89 | 17.71 |
| 160 | 15.09 | 14.93 | 15.12 | 14.98 |
| 165 | 12.64 | 12.47 | 12.67 | 12.58 |
| 167 | 11.74 | 11.57 | 11.77 | 11.70 |
| 170 | 10.48 | 10.32 | 10.52 | 10.48 |
| 175 | 8.622 | 8.458 | 8.656 | 8.671 |
| 180 | 7.031 | 6.874 | 7.064 | 7.127 |
| 185 | 5.686 | 5.539 | 5.717 | 5.823 |
| 190 | 4.562 | 4.428 | 4.591 | 4.731 |

*Source: author*

Table 7b: **Example 2 – comparison of absolute errors**

| Strike | Vorst-lognormal | Vorst-Gentle | Lognormal |
|---|---|---|---|
| 140 | 0.19 | 0.09 | 0.21 |
| 145 | 0.19 | 0.07 | 0.21 |
| 150 | 0.17 | 0.03 | 0.20 |
| 155 | 0.15 | 0.01 | 0.18 |
| 160 | 0.11 | 0.06 | 0.14 |
| 165 | 0.06 | 0.11 | 0.09 |
| 167 | 0.04 | 0.13 | 0.07 |
| 170 | 0.00 | 0.16 | 0.04 |
| 175 | 0.05 | 0.21 | 0.01 |
| 180 | 0.10 | 0.25 | 0.06 |
| 185 | 0.14 | 0.28 | 0.11 |
| 190 | 0.17 | 0.30 | 0.14 |
| **Mean** | **0.11** | **0.14** | **0.12** |
| **Standard dev.** | **0.06** | **0.10** | **0.07** |

*Source: author*

range of strike prices. In this table, the at-the-money strike is 167. The analytical models tend to overestimate the time value for in-the-money options and underestimate the time value for out-of-the-money strike values. Table 6b (page 57) gives a further comparison of the absolute errors between the analytical models and the Monte Carlo option values. These errors can also be thought of as errors in the analytical estimation of the options' time value. The last two rows of the table display the sample mean and standard deviation of the absolute errors across the range of strike values. These estimates can be used to compare the accuracy of the analytic models. The hybrid Vorst-lognormal and pure lognormal models exhibit slightly better accuracy than the Vorst-Gentle model. The accuracy of all three models degrades as the option moves further out-of-the-money.

### *Example 2: Unequal number of fixings*

In our second example, the basket items have different numbers of reset dates, as was shown in table 5 (page 56). All other parameters for this option are identical to those in example 1. This option is more difficult to price, because of the varying number of reset dates per basket item. The hybrid models can be easily extended to include these situations, because an adjusted volatility is computed for each basket item independently, based on the specification of its reset dates.

Table 7a shows a comparison of the approximate model values against Monte Carlo simulation. As observed in example 1, the time value of the in-the-money options is generally overestimated by the models, while the time value of the out-of-the-money options is underestimated. Table 7b gives the absolute errors when compared to the Monte Carlo values. The Vorst-lognormal and the lognormal models yield about the same accuracy, with the Vorst-Gentle model indicating somewhat larger errors.

### *Model behaviour with correlation*

We also selectively studied the behavior of the Asian basket option models using different correlation assumptions. We let the off-diagonal correlations of table 2 vary from 0 to 1. The model values are shown in tables 8 and 9, while the corresponding absolute errors are plotted in figures 4 and 5. The hybrid Vorst-Gentle model shows the most variation in model error as a function of the correlation. A thorough study of the dependence of model valuation in terms of correlation or other parameters is beyond the scope of this paper.

## Asian basket spread option model validation

### *Example 3: equal number of fixings.*

Table 10 gives the parameters associated with the Asian basket spread option. The reset dates are identical to those of example 1 (see table 4). The main difference between Asian basket options and Asian basket spread options is that the weights can have different signs depending on whether a basket item is bought or sold – corresponding to long and short positions.

All three analytical models become hybridised for this option. The option is divided into a buy- and a sell-side basket, and the option value is then calculated from a two-asset spread model. In our examples, we have used the Pearson spread option model (see Pearson (1995)) to evaluate the final value of the Asian basket spread options. All the parameters are identical to those of example 1, except that the weights of basket items 3, 4 and 5 are negative.

Table 8: **Example 1 – Asian basket option value behaviour with correlation**

| Corr | Vorst-logN | Vorst-Gentle | LogN | Monte Carlo |
|---|---|---|---|---|
| 0 | 7.317 | 7.050 | 7.372 | 7.275 |
| 0.25 | 9.371 | 9.147 | 9.405 | 9.331 |
| 0.5 | 11.06 | 10.87 | 11.08 | 11.00 |
| 0.75 | 12.52 | 12.38 | 12.55 | 12.45 |
| 1 | 13.84 | 13.74 | 13.88 | 13.73 |

*Source: author*

Table 9: **Example 2 – Asian basket option value behaviour with correlation**

| Corr | Vorst-LogN | Vorst-Gentle | LogN | Monte Carlo |
|---|---|---|---|---|
| 0 | 6.497 | 6.310 | 6.559 | 6.483 |
| 0.25 | 8.276 | 8.118 | 8.355 | 8.291 |
| 0.5 | 9.740 | 9.608 | 9.842 | 9.771 |
| 0.75 | 10.98 | 10.87 | 11.11 | 10.96 |
| 1 | 12.16 | 12.09 | 12.32 | 12.22 |

*Source: author*

Table 10: **Asian basket spread option parameters**

| | ATM Strike | Disc factor | Time | No of items | Call/put |
|---|---|---|---|---|---|
| | 3.0 | 0.942539 | 0.98630 | 5 | 1 |
| | **Item1** | **Item 2** | **Item 3** | **Item 4** | **Item 5** |
| **Prices** | 50.00 | 35.00 | 38.00 | 19.00 | 25.00 |
| **Weights** | 1.0 | 1.0 | (1.0) | (1.0) | (1.0) |
| **Volatility** | 0.50 | 0.25 | 0.35 | 0.45 | 0.10 |

*Source: author*

| Strike | Vorst-LogN | Vorst-Gentle | LogN | Monte Carlo |
|---|---|---|---|---|
| –25 | 26.49 | 26.48 | 26.49 | 26.53 |
| –20 | 21.92 | 21.92 | 21.93 | 21.95 |
| –10 | 13.48 | 13.46 | 13.49 | 13.43 |
| –5 | 9.888 | 9.864 | 9.906 | 9.809 |
| 0 | 6.904 | 6.876 | 6.926 | 6.834 |
| 3 | 5.433 | 5.405 | 5.457 | 5.387 |
| 5 | 4.585 | 4.557 | 4.610 | 4.560 |
| 10 | 2.903 | 2.878 | 2.928 | 2.936 |
| 15 | 1.760 | 1.739 | 1.782 | 1.840 |
| 20 | 1.026 | 1.011 | 1.044 | 1.134 |
| 25 | 0.5791 | 0.5684 | 0.5927 | 0.6918 |

Table 11a: **Example 3 – comparison of option values for Asian basket spread option**

*Source: author*

| Strike | Vorst-LogN | Vorst-Gentle | LogN |
|---|---|---|---|
| –25 | 0.04 | 0.05 | 0.04 |
| –20 | 0.02 | 0.03 | 0.02 |
| –10 | 0.05 | 0.03 | 0.06 |
| –5 | 0.08 | 0.06 | 0.10 |
| 0 | 0.07 | 0.04 | 0.09 |
| 3 | 0.05 | 0.02 | 0.07 |
| 5 | 0.02 | 0.00 | 0.05 |
| 10 | 0.03 | 0.06 | 0.01 |
| 15 | 0.08 | 0.10 | 0.06 |
| 20 | 0.11 | 0.12 | 0.09 |
| 25 | 0.11 | 0.12 | 0.10 |
| **Mean** | **0.06** | **0.06** | **0.06** |
| **Standard dev.** | **0.03** | **0.04** | **0.03** |

Table 11b: **Example 3 – comparison of absolute errors**

*Source: author*

| Strike | Vorst-LogN | Vorst-Gentle | LogN | Monte Carlo |
|---|---|---|---|---|
| –25 | 26.59 | 26.58 | 26.59 | 26.60 |
| –20 | 22.11 | 22.10 | 22.12 | 22.10 |
| –10 | 13.93 | 13.91 | 13.97 | 13.86 |
| –5 | 10.48 | 10.45 | 10.53 | 10.40 |
| 0 | 7.595 | 7.558 | 7.648 | 7.520 |
| 3 | 6.149 | 6.111 | 6.204 | 6.095 |
| 5 | 5.303 | 5.266 | 5.359 | 5.268 |
| 10 | 3.579 | 3.545 | 3.633 | 3.596 |
| 15 | 2.343 | 2.314 | 2.390 | 2.404 |
| 20 | 1.494 | 1.471 | 1.533 | 1.585 |
| 25 | 0.9319 | 0.9141 | 0.9627 | 1.037 |

Table 12a: **Example 4 – comparison of option values for Asian basket spread option**

*Source: author*

| Strike | Vorst-LogN | Vorst-Gentle | LogN |
|---|---|---|---|
| –25 | 0.01 | 0.02 | 0.01 |
| –20 | 0.01 | 0.01 | 0.02 |
| –10 | 0.07 | 0.04 | 0.11 |
| –5 | 0.09 | 0.05 | 0.13 |
| 0 | 0.07 | 0.04 | 0.13 |
| 3 | 0.05 | 0.02 | 0.11 |
| 5 | 0.03 | 0.00 | 0.09 |
| 10 | 0.02 | 0.05 | 0.04 |
| 15 | 0.06 | 0.09 | 0.01 |
| 20 | 0.09 | 0.11 | 0.05 |
| 25 | 0.10 | 0.12 | 0.07 |
| **Mean** | **0.06** | **0.05** | **0.07** |
| **Standard dev.** | **0.03** | **0.04** | **0.05** |

Table 12b: **Example 4 – comparison of absolute errors**

*Source: author*

Unlike the analytical models, the Monte Carlo simulations do not have inherent difficulties dealing with weights of different signs.

Table 11a compares the option values to those of the Monte Carlo simulation. For this Asian basket spread option, the at-the-money strike is 3. For this option, the time value of both the in-the-money and out-of-the-money options is slightly underestimated. Table 11b shows a comparison of the absolute errors relative to the Monte Carlo values. The biggest absolute errors tend to occur for out-of-the-money strike values.

### *Example 4: unequal number of fixings*

For this option, the reset dates are to those of example 2 (see table 5). All other option parameters are identical to the previous example. Table 12a provides the option values while table 12b gives the absolute errors relative to Monte Carlo for this Asian basket spread option. The errors are small for options struck in-the-money. Out-of-the-money options tend to be undervalued.

## Conclusions

Valuation of Asian basket or Asian basket spread options using conventional Monte Carlo methods can be very computationally intensive, because of their path-dependent nature. This makes it unfeasible for practitioners. Hence, it is vital that analytical models be employed to value these options approximately.

We tested three analytical averaging option models for accuracy in several examples for a range of strikes. The tests were intended to stress the option models across a relatively wide range of strike values from in-the-money to out-of-the-money conditions. All the analytical models yield fairly accurate results for both Asian basket and Asian basket spread models, even in cases where where each basket item had an unequal number of reset dates per item.

As the options became deep out-of-the-money, the models exhibited some degradation in accuracy compared to Monte Carlo simulations. EPRM

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