Estimating economic capital allocations for market and credit risk

Paul Kupiec
Federal Deposit Insurance Corporation, 550 17th Street NW, Washington DC 20429, USA

Value-at-Risk (VAR) measures often are used as a basis for setting so-called "economic capital" or buffer stock measures of equity capitalization requirements. VAR measures do not account for the time value of money or the equilibrium required return premium for credit risk on a firm's funding debt, and consequently they produce biased estimates of economic capital. The bias in common VAR approaches increases with the horizon and consequently is most pronounced in the credit risk setting where capital allocations horizons typically coincide with extended holding periods, but the bias is also important in the market risk setting when capital allocations are set for similar tenors. Accurate capital estimates can be obtained using a VAR-like measure that is constructed relative to a portfolio's initial market value and augmented by an estimate of the interest compensation required on funding debt.

1 Introduction

Value at risk (VAR) measures are widely employed to estimate equity capital requirements or so-called "economic capital." In the buffer stock context, economic capital is the equity financing that is required in a firm's capital structure to ensure that that the default rate on a firm's funding debt never exceeds a maximum optimal target rate selected by a firm's management. In the market risk setting, banking regulation, industry practice, and the risk management literature often equate market risk economic capital with a specific VAR measure of market risk or perhaps a multiple thereof.[1] In the context of credit risk, firms typically set credit risk capital equal to unexpected credit loss, the measure of credit risk typically estimated in credit VAR models.[2]

The widespread use of VAR models to estimate capital requirements is a testament to the intuitive appeal of VAR measures. It may come as a surprise to learn that the simplistic intuition that underlies a VAR approach for capital allocation has serious shortcomings in both the market risk and credit risk settings. Perfectly accurate VAR models produce biased estimates of equity capital requirements because they ignore the time value of money and the equilibrium risk compensation that is required by investors.

In the context of a rigorous equilibrium model of firm capital structure, this

The views expressed in this paper represent those of the author, not the FDIC.
[1] See, for example, Wilson (1997), Smithson (1997), or the discussion in Kupiec (1999).
[2] See, for example, the survey results reported in The Basel Committee on Banking Supervision, "Credit risk modeling: Current practices and applications," April 1999, p. 13.

paper constructs accurate buffer stock capital allocations for both the market and credit risk settings. These equity capital funding requirements differ from those recommended by the traditional VAR capital allocation procedures in two important ways. One difference is in the construction of the VAR measure. When used for capital allocation purposes, VAR must be measured relative to a portfolio's initial market value.[3] The second required adjustment is to augment the modified VAR measure with an estimate of the equilibrium interest payments that must be made on the firm's funding debt. This recipe for calculating unbiased buffer stock capital measures holds for both market and credit risk capital allocations.

In the market risk setting, it is common to calculate VAR measures for an entire trading portfolio. The portfolio may include short and long positions and derivative or other structured positions that create both asset and liability positions for the firm. Positions in the trading portfolio represent counterparty payment or receipt obligations that arise from securities and derivative contracts. VAR methods focus on the net mark-to-market values produced by the asset and liability positions in the portfolio, and VAR estimates are often used to set buffer stock equity capital requirements for trading operations.

Market risk VAR calculations on trading portfolios typically exclude the balance sheet debt and equity securities issued by the firm itself to fund its investments and trading operations. They treat the net value of trading portfolio positions as an asset and estimate the capital requirements necessary to fund this composite investment within the firm's maximum default rate targets. VAR-based capital measures for trading portfolios are subject to the same biases that arise in the simple case of allocating capital for a single long position in an equity or risky debt investment.

The results of this study show that augmented VAR-like measures can be used to set accurate buffer stock equity capital allocations, but the use of these measures may require a recalibration of thinking, especially in the context of credit risk capital allocation. Capital allocation VAR measures are not measures of credit risk. The credit VAR measure that is appropriate for capital allocation purposes measures loss relative to initial value, not relative to a promised payment stream, and so these VAR measures may be negative. Negative VAR measures are likely to be counterintuitive to many risk managers, and their usefulness may be restricted to capital allocation decisions.

An outline of the paper follows. Section 2 formally defines market risk and credit risk VAR measures. Section 3 discusses the flaw in the logic that underlies the common explanation that is used to support VAR-based approaches for capital allocation. Section 4 discusses the accurate construction of buffer stock capital allocations in the context of the Black and Scholes (1973) and Merton (1974) (BSM) model. Section 5 provides explicit examples of alternative capital allocation calculations. Section 6 concludes the paper.

---

[3] Most discussions define a VAR measure relative to the mean of the end-of-period value (or return) distribution.

2 Defining a VAR measure

VAR is commonly defined to be the loss amount that could be exceeded by at most a maximum percentage of all potential future asset or portfolio value realizations at the end of a given time horizon.[4] Typically, the loss in a VAR measure is defined to be the difference between the expected value of a portfolio's future value (or return) distribution and a specific left-hand critical value of a potential profit and loss distribution. By convention, the loss in VAR calculations is reported as a positive value.

2.1 Market risk VAR

The "textbook" formulation for a market risk VAR assumes that assets' returns are normally distributed over the interval of interest. They measure loss relative to the expected value of the end-of-period profit and loss distribution. Let V_0 represent the present market value of the asset or portfolio. Assume the asset promises one-period returns, $\tilde{r}$, that are normally distributed with a mean of μ and a variance of σ^2. As it is commonly defined, a single-period VAR measured at the 99% coverage level, $\text{VAR}^{\mu}(0.99)$, is given by,

$$\text{VAR}^{\mu}(0.99) = V_0\left(2.33\sigma\right) \tag{1}$$

where the notation includes the superscript μ to designate that losses are measured relative to the mean of the distribution. Equation (1) also defines a 99% coverage VAR measure when the mean of the return distribution is assumed to be zero. $\text{VAR}^{\mu}(0.99)$ provides an estimate of a loss amount that will not be exceeded in 99% of all sample outcomes. Alternatively, the $\text{VAR}^{\mu}(0.99)$ loss threshold will be exceeded by at most 1% of all distribution outcomes. VAR^{μ} is the measurement basis for the Basel Internal Models Approach for setting market risk capital requirements.

If losses are measured relative to an asset's initial market value (instead of its average value), the 99% coverage VAR for an asset with normally distributed returns with a non-zero mean of μ and a variance of σ^2 is given by,

$$\text{VAR}^{V_0}\left(0.99\right) = V_0\left(2.33\sigma - \mu\right) \tag{2}$$

In typical short-horizon market risk applications that assume normally distributed returns, VAR^{μ} is used to monitor changes in trading portfolio risks typically using a one-day horizon. In this application, μ is either approximately zero or intentionally set to zero to minimize the effects of errors associated with the estimation of short-horizon expected returns, and there is little difference between the measures in (1) and (2). Capital allocation decisions, however, require VAR cal-

[4] This definition can be found *inter alia* in Duffie and Pan (1997), Hull and White (1998), Jorion (1995, 1997), Beder (1995), and Marshall and Siegel (1997).

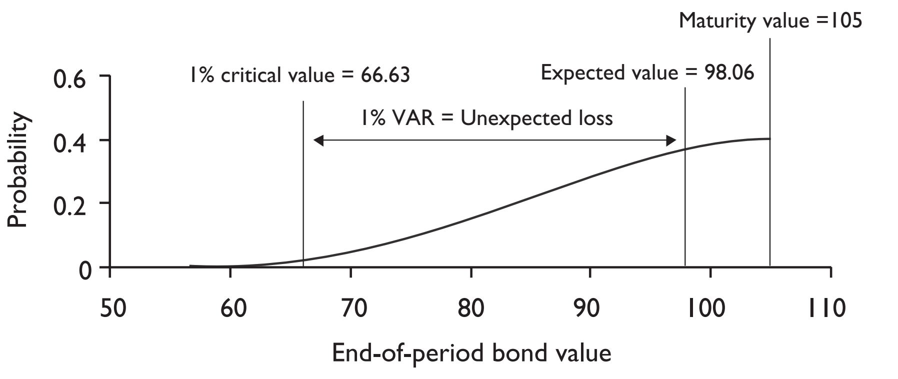
culations for holding periods substantially longer than a day. As the time horizon lengthens, the differences in measures (1) and (2) can become substantial.[5]

2.2 Credit risk VAR

In the credit risk setting, VAR techniques were developed to measure risks over relatively long horizons primarily for use in capital allocation and RAROC[6] decisions. In this setting, it has been widely presumed that an appropriate approach for setting the equity share of funding for a credit portfolio is to set equity capital equal to an estimate of a portfolio's so-called unexpected credit loss. Unexpected credit loss is defined as the difference between the mean of the end-of-period credit risk profit and loss distribution and the loss associated with a user-selected critical value in the loss tail of the distribution. Unexpected credit loss is the credit risk profit-and-loss distribution counterpart to a VAR^{μ} measure.[7]

A stylized credit VAR unexpected credit loss measure is illustrated in Figure 1. The probability distribution pictured represents the true probabilities associated with all potential end-of-period values that may be realized on an asset (portfolio) with credit risk. The potential profit and loss distribution of interest is generated by potential changes in the value of credit risk sensitive exposures to individual counterparties over the horizon that has been selected to measure credit risk and set capital.[8] Credit VAR models attempt to estimate unexpected credit losses in either

FIGURE 1 Credit VAR calculation.



Stylized 1% unexpected credit loss calculation over a one-year horizon for a discount bond with a maturity of 1 year and a par value of 105.

[5] Kupiec (1999) provides additional discussion.
[6] RAROC is the acronym for risk-adjusted return on capital.
[7] This definition of unexpected loss appears, for example, in the *CreditMetrics Technical Document* (1996), Wilson (1997), Saunders (1999), and "Credit risk modeling: Current practices and applications" (1999).
[8] See The Basel Committee on Banking Supervision (1999) or Saunders (1999) for a discussion of alternative approaches for estimating the end-of-period value distribution in alternative credit VAR models.

a mark-to-market or a held-to-maturity setting. If the asset has yet to mature in the horizon of interest, the end-of-period value distribution represents the asset's potential mark-to-market values or its range of potential values in an early default. If the end of the period in question corresponds to the maturity of the asset, the variation in end-of-period values owes entirely to variation in default severity.

In Figure 1, the 99% coverage credit VAR^{μ} measure (unexpected credit loss measure) is 31.43 – the difference between the expected end of period value of the bond, 98.06, and the 1% critical value from the left-hand tail of the distribution, 66.63. In sample outcomes, 99% of all credit losses (relative to the expected payout) will be contained within this unexpected credit loss measure.

3 Unbiased buffer stock capital allocation

A buffer stock capital allocation is the equity portion of a funding mix that can be used to finance an asset (portfolio) with the maximum amount of debt finance subject to a maximum acceptable probability of default on the funding debt.[9] The intuition that underlies the calculations required to estimate an unbiased buffer stock capital allocation is transparent when considering capital allocation on portfolios composed of long positions in traditional financial assets such as bonds or equities because the value of the portfolio cannot go below zero; that is, the maximum value that can be lost is the current market value of the portfolio. If portfolio losses can exceed the initial market value of a portfolio, as they can for example when a portfolio includes short positions, futures, derivatives, or other structured products, then a buffer stock capital allocation problem may include additional complications. Such portfolios may require 100% equity funding, and yet the *ex ante* default rate on the firm's implicit liability commitments may exceed the management selected target rate. When the admissible amount of debt finance is zero, the capital allocation objective function is focused on controlling the probability of default on the liabilities that are created through short portfolio positions, derivatives, and other structured contracts.

If potential losses may exceed the market value of the portfolio with significant probability, capital in excess of the initial market value of the current portfolio's positions may be required to achieve the target default rate on the firm's future payment obligations. In such instances, the capital allocation need not consider required interest payments on the firm's funding debt (which are 0), but assumptions are required regarding how the required additional equity capital buffer is to be invested in the portfolio. The additional equity necessary to limit the probability of default varies depending on how the additional equity investment is used to augment the portfolio. The required equity injection differs, for example, if the additional equity is invested in Treasury obligations rather than in additional equity or defaultable bonds. We consider this issue in greater detail later.

[9] We make no claim that this objective function formally defines a firm's optimal capital structure – indeed it almost certainly does not. It is, however, the objective function that is consistent with VAR-based capital allocation schemes and an approach commonly taken by banks according to the Basel Committee on Banking Supervision's (1999) survey results.

3.1 Capital allocation for long risky bond or equity investments

Consider the use of a 99%, one-year measure, $VAR^{V_0}(0.99)$, to determine the necessary amount of equity funding for a long bond or equity position under a buffer stock approach for setting equity capital. By definition, there is less than a 1% probability that the asset's value will ever post a loss that exceeds its $VAR^{V_0}(0.99)$ measure. That is, if the amount of equity financing in the capital structure is set equal to $VAR^{V_0}(0.99)$, the implication is that there is less than a 1% chance that any investment portfolio loss will ever exceed the value of the firm's initial equity. The interpretation that underlies VAR capital allocation techniques is that, by setting equity financing in the amount $VAR^{V_0}(0.99)$ there is at most a 1% chance that the firm will default on its debt. This intuition, however appealing, is flawed.

Assume that VAR will be measured from the asset's initial market value and that VAR measures are statistically accurate. In this case VAR can never exceed V_0. If the firm were to set the share of equity funding equal to $VAR^{V_0}(0.99)$, the amount of debt finance required to fund the asset would be $V_0 - VAR^{V_0}(0.99)$. If the firm borrows $V_0 - VAR^{V_0}(0.99)$, it must however pay back more than $V_0 - VAR^{V_0}(0.99)$ if it is to avoid default. An unbiased buffer stock capital allocation rule for a 1% target default rate is to set equity capital equal to $VAR^{V_0}(0.99)$ plus the interest that will accrue on the funding debt issue. This capital allocation recipe holds for both market and credit risks.

3.2 Capital allocation when losses may exceed initial portfolio value

When a portfolio includes short or derivative positions, losses at the VAR horizon may exceed the portfolio's initial market value. When VAR estimates are measured relative to a portfolio's initial market value, and the target VAR estimate exceeds the initial market value of the portfolio, V_0, the portfolio's risk characteristics do not permit debt financing under the target default rate objective and the portfolio must be 100% equity financed. Even then, the default rate on the firm's implicit liabilities will exceed the optimal target rate used in the firm's capital allocation rules.

If the management's objective is to contain the default probability on all of the firm's liabilities to within a target rate α, when $VAR^{V_0}(1-\alpha) > V_0$, the firm needs additional equity capital to satisfy its objectives. How much additional equity depends on how the new capital injection is invested. The most straightforward way to satisfy the target default rate is to add risk-free bonds to the portfolio. Since these bonds accrue interest at the risk-free rate (r_f) over the VAR horizon, the amount of bonds that must be added to the portfolio to achieve the default rate target is,[10]

$$e^{-r_f}\left(VAR^{V_0}(1-\alpha) - V_0\right) \tag{3}$$

[10] It is implicitly assumed that the risk-free rate is for a tenor equal to the VAR horizon.

If risk-free bonds are not the preferred asset class to use for reducing the implicit default rate on the portfolio's liabilities, the capital allocation problem becomes an iterative process. Risky equities, defaultable bonds, or other risky assets must be added to the portfolio until

$$\mathrm{VAR}^{V_0'}(1-\alpha) - V_0' = 0 \tag{4}$$

where V_0' indicates that the portfolios' initial market value after the addition of new risky assets and $\mathrm{VAR}^{V_0'}(1 - \alpha)$ measures the augmented portfolio's VAR.

The upshot of this analysis is that if one uses the correct VAR measure – one in which the VAR's right-side boundary is set by the asset's initial market value VAR^{V_0} – and the VAR^{V_0} estimate is augmented by the interest payments that will be required by investors who purchase the funding debt, the VAR methodology can provide accurate measures of buffer stock capital. This is true in both the market risk and the credit risk setting. The required VAR^{V_0} calculation, while modified by comparison with many discussions, is a straightforward modification. The complication is introduced by the necessity of obtaining estimates of the required interest payments on funding debt. The following section describes the capital allocation process in the context of a specific equilibrium asset pricing model that will allow for the determination of the required interest payments on funding debt.

4 Unbiased buffer stock capital allocation in a Black–Scholes–Merton model

Under various simplifying assumptions,[11] Merton (1974) established that the Modigliani–Miller capital structure irrelevance theorem holds in the presence of risky debt. That is, the market value of the firm is completely independent of capital structure and the probability of default can be chosen freely by management.

If the risk-free term structure is flat and if a firm issues only pure discount debt and asset values follow geometric Brownian motion, Black and Scholes (1973) and, independently, Merton (1974) (hereafter BSM) have demonstrated that the market value of a firm's debt issue is equal to the market value the issue would have if it were default free, less the market value of a Black–Scholes put option written on the value of the firm's assets. The put option has a maturity equal to the maturity of the debt issue and strike price equal to the par value of the discount debt. If B_0 represents the bond's initial equilibrium market value and *Par* represents its promised payment at maturity date M, the BSM model requires

$$B_0 = Par\ \mathrm{e}^{-r_f M} - Put\left(A_0, Par, M, \sigma\right) \tag{5}$$

[11] There are no taxes, transactions are costless, short sales are possible, trading takes place continuously if borrowers and savers have access to the debt market on identical risk-adjusted terms, and investors in asset markets act as perfect competitors.

where $Put(A_0, Par, M, \sigma)$ represents the value of a Black–Scholes put option on an asset with an initial value of A_0, a strike price of *Par*, a maturity of *M*, and an instantaneous return volatility of σ. The default (put) option is a measure of the credit risk of the bond. While Merton (1974) shows that the model will generalize (as to term structure assumptions, coupon payments, and generalized volatility assumptions), the capital allocation discussion that follows will be based upon the simplest formulation of the BSM model.[12]

4.1 Market risk capital

In the BSM model, the firm's underlying assets evolve in value according to geometric Brownian motion and have future values that exhibit so-called "market risk" in the vernacular of risk managers. In this setting, the firm's market risk capital allocation problem involves the selection of the firm's debt-equity funding mix under an objective of achieving a target default rate on its funding debt. In the market risk setting, the VAR calculation is applied to the physical probability distribution for the assets' value at a horizon equal to the desired maturity of the firm's funding debt.

Under the assumptions of the BSM model, the value of the firm's assets evolve following,

$$\mathrm{d}A = \mu\, A\, \mathrm{d}t + \sigma\, A\, \mathrm{d}t \tag{6}$$

where $\mathrm{d}z$ is a standard Wiener process. If A_0 represents the initial value of the firm's assets, and A_T the value of the firm's assets at time *T*, Itô's lemma implies

$$\ln A_T - \ln A_0 \sim \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right] \tag{7}$$

where $\phi[a, b]$ represents the normal density function with a mean of "*a*" and a standard deviation of "*b*". Equation (7) defines the physical probability distribution for the end-of-period value of the firm's assets,

$$\tilde{A}_T \sim A_0 \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\tilde{z}} \tag{8}$$

where $\tilde{z} \sim \phi[0, 1]$.

Let $\Phi(x)$ represent the cumulative density function for a standard normal random variable evaluated at x, and $\Phi^{-1}(\alpha)$, the inverse of this function evaluated at $0 \le \alpha \le 1$. The market risk $\mathrm{VAR}^{V_0}(1 - \alpha)$ measure that is appropriate for calculating an equity capital allocation consistent with a target default rate of α for a funding debt maturity of *T*, is given by

[12] That is, it assumed that the term structure is flat, asset volatility is constant, the underlying asset pays no dividend or convenience yield, and all debt securities are pure discount issues.

$$\mathrm{VAR}^{V_0}\left(1-\alpha\right)=A_0\left[1-\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)}\right] \tag{9}$$

$$A_0-\mathrm{VAR}^{V_0}\left(1-\alpha\right)=A_0\,\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \tag{10}$$

is the maximum par value of discount debt that the firm can issue without violating its target default rate. The BSM debt pricing condition (Expression (5)) can be used to determine the initial market value of this debt issue, $B_0^{\max}$. The difference between the par value and the initial market value of the debt is the equilibrium interest compensation that must be offered to the firm's debt holders. In the BSM model setting, the interest payment is,

$$\begin{aligned}&A_0-\mathrm{VAR}\left(1-\alpha\right)-B_0^{\mathrm{Max}}=\\&A_0\left[\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)}-\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)-r_fT}\right]\\&+Put\left(A_0,A_0\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)},T,\sigma\right)\end{aligned} \tag{11}$$

Expression (11) represents the interest amount that must be added to $\mathrm{VAR}^{V_0}(1-\alpha)$ to calculate the equity capital allocation needed to achieve the target default rate on funding debt. The true amount of equity, E_0, required to achieve a target default rate of α on funding debt of maturity T, is given by

$$\begin{aligned}&A_0\left[1-\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)-r_fT}\right]\\&+Put\left(A_0,A_0\mathrm{e}^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}\,\Phi^{-1}(\alpha)},T,\sigma\right)\end{aligned} \tag{12}$$

The components of the equity capital allocation are instructive. The first component of Expression (12) increases equity over $\mathrm{VAR}^{V_0}(1-\alpha)$ to allow funding debt holders to receive a risk-free return on their investment. The second term further increases equity capital to ensure that the funding debt holders receive the proper credit risk interest spread on their investment.

4.2 Credit risk capital

In order to illustrate the buffer stock capital allocation technique that is appropriate for assets with credit risk, it necessary to introduce a modified version of the

BSM model in order to value the funding debt of a firm that purchases credit risky assets. Consider the case in which a firm's only asset is a risky BSM discount debt issued by another firm. Assume that the firm will fund this bond with its own discount debt and equity issues. In this setting, the firm's funding debt issue is a compound option.

Let $\tilde{A}_T$ represent the time T value of the assets that support the purchased discount debt. Let Par_P represent the par value of the purchased discount bond and Par_F represent the par value of the discount bond that is used to fund the asset purchase. If the maturity of the firm's funding debt matches the maturity of the firm's asset (both equal to M), then the end-of-period cashflows that accrue to the firm's debt holders are given by,

$$\min\left[\min\left(\tilde{A}_M, Par_P\right), Par_F\right] \tag{13}$$

If the funding debt is of a shorter maturity (T) than the purchased discount bond (M), then the end-of-period cashflows that accrue to the firm's funding debt holders are given by,

$$\min\left[\left(Par_P\, \mathrm{e}^{-r_f(M-T)} - Put\left(\tilde{A}_T, Par_P, M-T, \sigma\right)\right), Par_F\right] \tag{14}$$

Equilibrium absence of arbitrage conditions impose restrictions on the underlying asset's Brownian motion's drift term, $\mu = r_f + \lambda\sigma$, where λ is the market price of risk associated with the firm's assets. Define $\mathrm{d}A^{\eta} = (\mu - \lambda\sigma)A^{\eta}\mathrm{d}t + A^{\eta}\sigma\mathrm{d}z$ to be the "risk-neutralized" process that is used to value derivative claims after an equivalent martingale change of measure. The probability distribution of the underlying end-of-period asset values after the martingale change of measure, $\tilde{A}_M^{\eta}$, is

$$\tilde{A}_T^{\eta} \sim A_0\, \mathrm{e}^{\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\tilde{z}} \tag{15}$$

When the maturity of the firm's funding debt matches the maturity of the firm's asset (both equal to M), the equivalent martingale probability distribution of the end-of-period asset's value, $\tilde{A}_M^{\eta}$, is used to calculate the initial market value of the funding discount bond by discounting (at the risk-free rate) the expected value of (13) taken with respect to probability density of $\tilde{A}_M^{\eta}$.[13] In the alternative case in which the funding debt is of a shorter maturity (T) than the purchased discount bond (M), the initial equilibrium value of the funding debt is the discounted value (at the risk-free rate) of the expected value of Expression (14), where the expectation is taken with respect to the equivalent martingale probability density $\tilde{A}_T^{\eta}$.

[13] Alternatively, Geske (1977, 1979) provides a closed-form expression for the value of the compound option.

Given the equilibrium valuation relationship that must be satisfied by the firm's funding debt, we now consider the buffer stock capital allocation process for assets with credit risk. Assume that the firm's objective is to maximize the use of debt funding subject to limiting the default rate on its funding debt to a maximum acceptable rate. Recall that the firm's investment asset is a BSM risky discount bond of maturity *M*. We consider initially capital allocation when the maturity of the funding debt is equal to the maturity of the purchased bond.

4.3 Held-to-maturity credit VAR

At maturity, the payoff of the firm's purchased bond is given by $\min[Par_P, \tilde{A}_M]$. The credit risk VAR measure appropriate for credit risk capital allocation is given by

$$\begin{aligned} &\mathrm{VAR}^{V_0}_{\mathrm{Credit}}(1-\alpha) \\ &= B_0 - \min\left[Par_P, A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \right] \end{aligned} \tag{16}$$

where B_0 is the initial market value of the purchased discount debt given by Expression (5), and α is the target default rate on the funding debt. If α is sufficiently small (which will be assumed), the expression

$$\min\left[Par_P, A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \right] \tag{17}$$

simplifies to

$$A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \tag{18}$$

and consequently, the expression for credit $\mathrm{VAR}^{V_0}_{\mathrm{Credit}}(1-\alpha)$ is

$$\mathrm{VAR}^{V_0}_{\mathrm{Credit}}(1-\alpha, M, M) = B_0 - A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \tag{19}$$

The notation for the credit VAR measure includes three arguments, the target default rate α, the maturity of the funding debt issue, *M* (the second argument), and the maturity of the credit risky asset, *M*. The utility of this unusual notation will become clear in the subsequent section when we consider shorter horizons.

Similar to the market risk case,

$$B_0 - \mathrm{VAR}^{V_0}_{\mathrm{Credit}}\left(1-\alpha, M, M\right) = A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \tag{20}$$

determines the maximum par (maturity) value of the funding debt that can be issued consistent with the target default rate. The initial market value of this funding debt issue is given by

$$E^{\eta}\left[\min\left[\min\left(\tilde{A}_M, Par_P\right), A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \tag{21}$$

where the notation $E^{\eta}[\ ]$ denotes the expected value operator with respect to the probability density for $\tilde{A}^{\eta}_M$, the equivalent martingale probability density function of the underlying assets' future value. Using these relationships, the equilibrium required interest payment on the funding debt is given by

$$\begin{aligned} &A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)} \\ &\quad - E^{\eta}\left[\min\left[\min\left(\tilde{A}_M, Par_P\right), A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \end{aligned} \tag{22}$$

Expressions (16) and (22) imply that the initial equity allocation consistent with the target default rate α is given by,

$$B_0 - E^{\eta}\left[\min\left[\min\left(\tilde{A}_M, Par_P\right), A_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}\right]\right] e^{-r_f M} \tag{23}$$

4.4 Mark-to-market credit VAR

The credit VAR profit and loss distribution differs according to whether the horizon corresponds to the maturity of the credit risky asset or a shorter period of time. When calculating credit VAR for buffer stock capital purposes, the credit VAR horizon must be equal to the maturity of the funding debt issue. Any other credit VAR horizon will produce capital allocations with default rates that differ from the intended target.[14]

In a dynamic setting, when the end-of-period value of the purchased bond is less than the par value of a firm's outstanding and maturing funding debt, it may be possible for the firm to refinance its existing debt and avoid default without an equity injection. In such instances, however, the implied default rate on the new debt issue is necessarily much higher than the firm's original target default rate.

[14] For further discussion, see Kupiec (2001b).

In order to refinance, the firm must dilute its equity value by promising a greater share of the end-of-period cashflows to the new bond investors. Regardless of whether the firm is actually forced to default when the value of the purchased debt falls below the funding debt's par value at maturity, the firm's capital allocation objective has been violated and the firm cannot avoid default or continue in business operating at its optimal target default rate unless the shareholders inject new equity capital.

When the firm's funding debt matures at date T before the purchased risky discount bond's maturity, M, $T < M$ the $1 - \alpha$ level credit VAR is given by

$$\mathrm{VAR}^{V_0}_{\mathrm{Credit}}\left(1-\alpha, T, M\right) = B_0 - \left(Par_P\, e^{-r_f (M-T)} - Put\left(A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}, Par_P, M-T, \sigma \right)\right) \tag{24}$$

$B_0 - \mathrm{VAR}^{V_0}_{\mathrm{Credit}}(1 - \alpha, T, M)$ is used to determine the maximum par value of the funding debt that will satisfy the firm's target default rate. The maximum par value on the funding debt is given by

$$Par_F\left(\alpha, T, M\right) = Par_P\, \mathrm{e}^{-r_f (M-T)} - Put\left(A_0\, \mathrm{e}^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,\Phi^{-1}(\alpha)}, Par_P, M-T, \sigma \right) \tag{25}$$

where the arguments in the notation for $Par_F(\alpha, T, M)$ conform with those in $\mathrm{VAR}^{V_0}_{\mathrm{Credit}}(1 - \alpha, T, M)$.

Incorporating the expression for $Par_F(\alpha, T, M)$, the initial market value of the funding debt issue is

$$E^{\eta}\left[\min\left[\left(Par_P\, \mathrm{e}^{-r_f (M-T)} - Put\left(\tilde{A}_T, Par_P, M-T, \sigma\right)\right), Par_F\left(\alpha, T, M\right)\right]\right] \mathrm{e}^{-r_f T} \tag{26}$$

and the equilibrium required interest payment on the funding debt is given by

$$Par_F\left(\alpha, T, M\right) - E^{\eta}\left[\min\left[\left(Par_P\, \mathrm{e}^{-r_f (M-T)} - Put\left(\tilde{A}_T, Par_P, M-T, \sigma\right)\right), Par_F\left(\alpha, T, M\right)\right]\right] \mathrm{e}^{-r_f T} \tag{27}$$

Expressions (24) and (27) imply that the equity allocation consistent with a target default rate of α is given by

$$B_0 - E^{\eta}\left[\min\left[\left(Par_P\, e^{-r_f(M-T)} - Put\left(\tilde{A}_T, Par_P, M-T, \sigma\right)\right), Par_F\left(\alpha, T, M\right)\right]\right] e^{-r_f T} \quad (28)$$

4.5 Remarks

Equations (23) and (28) respectively are the equity capital allocations necessary to achieve the target default rates of the firm's funding debt in the held-to-maturity and the market-to-market credit risk cases. In both of these expressions, the equity capital requirement is determined by the α critical value of the risky discount debt's supporting asset distribution. The underlying capital allocation credit VAR measures (Expressions (19) and (24)) are not measures of the credit risk of the risky asset. Credit risk is defined relative to the promised maturity payment on a fixed income asset, not its initial value. The results demonstrate that an asset's credit risk is not directly relevant for setting its economic capital allocation. These results challenge the long-standing tradition of linking the processes of credit risk measurement and capital allocation. Moreover, they also highlight the importance of establishing an accurate estimate of the initial mark-to-market value for a credit portfolio. Such a task is often thought to be particularly complicated in the case of bank loans.

Another perhaps less apparent implication of the analysis is that capital allocation credit VAR measures (Expressions (19) and (24)) can be negative. This will happen, for example, when a firm's target default is lower than the default rate on the asset it is purchasing. Such negative credit VAR measures are fully appropriate in the capital allocation context. They are, however, likely to be counterintuitive for many risk mangers who typically expect to find a positive relationship between credit VAR measures, risk, and capital allocations.

5 Some examples

The capital allocation process for both market and credit risk is illustrated in the context of the BSM model. Each example is based on an underlying asset with an initial value of 100, an instantaneous return volatility of $\sigma = 0.20$, an instantaneous drift rate $\mu = 0.08$, a market price of risk $\lambda = 0.15$, and a risk-free rate of 5%. Under these assumptions, the asset's physical and risk neutral value distribution functions are given by

$$\tilde{A}_T \sim 100\, e^{\left(0.08 - \frac{0.2^2}{2}\right) T + 0.2\sqrt{T}\,\tilde{z}} \quad (29)$$

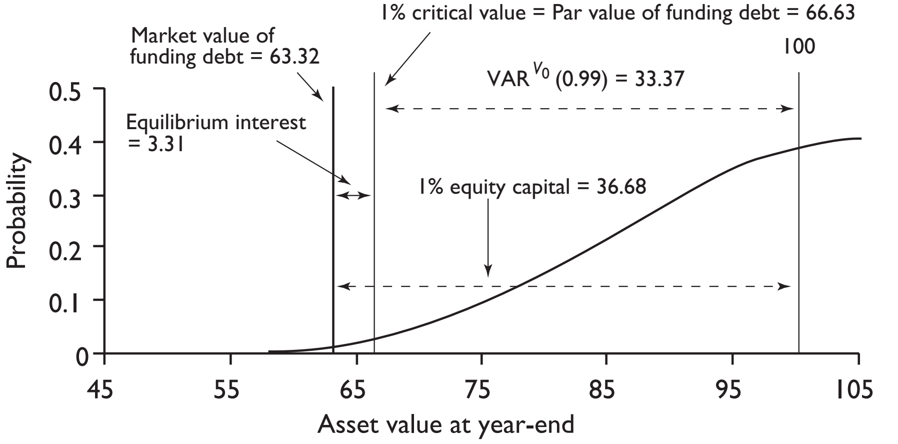
$$\tilde{A}_T^{\eta} \sim 100\, e^{\left(0.05 - \frac{0.2^2}{2}\right) T + 0.2\sqrt{T}\,\tilde{z}} \quad (30)$$

5.1 Long-horizon market risk capital allocation

The firm wishes to fund a risky asset with characteristics given by Expression (29) using the maximum amount of one-year discount debt possible subject to the constraint that the probability of default on the funding debt cannot exceed 1%. The probability distribution for the asset's value after one-year is pictured in Figure 2. The asset's future value is distributed lognormally, with a left-hand 1% critical value of 66.63. The one-year market risk VAR measure appropriate for capital allocation is 33.37, or the difference between the asset's initial market value (100) and the 1% critical value of its future value distribution, 66.63. The maximum par value of the funding debt that can be issued while remaining within the funding debt's target default rate of 1% is 66.63. Using the BSM equilibrium valuation equation for risky debt (Expression (5)), this discount debt issue will sell for 63.32 when issued. The difference between 66.63 and 63.32 is the equilibrium interest compensation required by the firm's funding debt holders (3.31). The equity capital requirement necessary to ensure that there is almost a 1% default rate on the firm's funding debt is equal to 36.68 – ie, the sum of $\text{VAR}^{V_0}(0.99) = 33.37$ and the equilibrium interest payment required by funding debt holders, 3.31.

For reference, consider the difference between the unbiased capital allocation, 36.38, and the capital allocation that would be estimated using a "traditional" long-horizon market risk VAR approach, when the mean of the return distribution is not assumed to be zero. If the equity capital allocation is set equal to the difference between the mean of the future value distribution, 108.33 (not shown in Figure 2), and the distribution's 1% critical value, 66.63, the estimated equity capital requirement is 41.70. In this example, the traditional VAR technique allocates 5.32 in unnecessary capital, or an excess of almost 15%.

FIGURE 2 Market risk capital allocation.



Market risk capital allocation example for an asset with an initial value of 100 and a future value that evolves according to geometric Brownian motion with an instantaneous drift rate of 8%, and an instantaneous return standard deviation of 20%. The risk-free rate is assumed to be 5%.

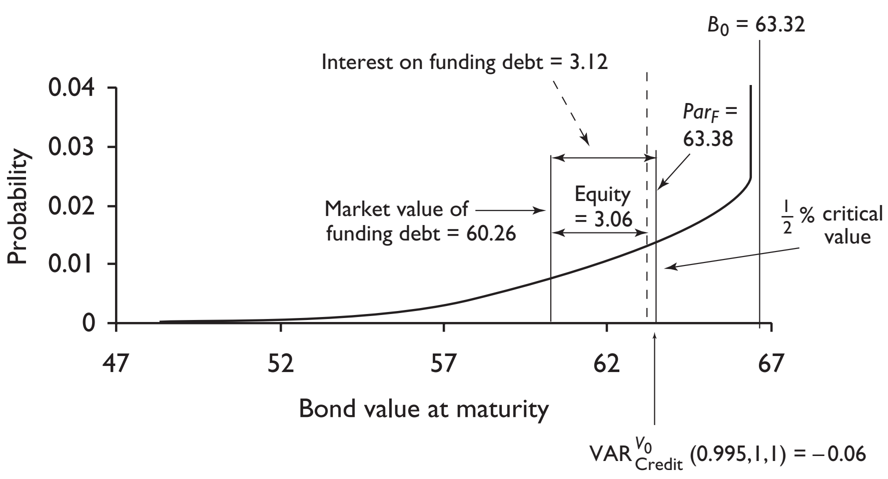
5.2 Held-to-maturity credit risk capital allocation

Figure 3 illustrates held-to-maturity capital allocation for a risky BSM discount bond that has a par value of 66.63 and an initial market value of 63.32. This BSM bond is supported by an asset with an initial market value of 100 and future values that evolve according to Expression (29). This discount bond is identical to the bond sold by the firm in the prior example. The third-party firm that is purchasing this discount bond is assumed to have no other assets.

If the acquiring firm were to fund this bond entirely with debt, the funding debt will have a probability of default identical to the acquired BSM bond (1%). If this bond acquisition is in part funded with equity, the funding debt's probability of default will be less than 1%.

Assume that the objective of the firm that purchases the risky BSM discount bond is to maximize the use of debt finance, subject to limiting the probability of default on its funding debt to 0.5%. The 0.5% critical value of the acquired BSM bond underlying asset's future value distribution is 63.38. Notice that the 0.5% critical value is 0.06 greater than the bond's initial market value. Under this target default rate objective, the capital allocation credit VAR measure, $\text{VAR}^{V_0}_{\text{Credit}}(0.995, 1, 1)$, is given by 63.32 – 63.38 = – 0.06, or *negative* 6 cents. The par value of the funding debt consistent with the 0.5% target default rate is 63.38. Using Expressions (21) and (30), the initial value of the funding debt can be calculated to be 60.26. The required interest payment on the funding debt is 3.12 (or 63.38

FIGURE 3 Held-to-maturity credit risk capital allocation.



Held-to-maturity credit risk capital allocation for a one-year BSM risky discount bond with a par value of 66.63 that is supported by assets that have an initial market value of 100 and future values that evolve according to geometric Brownian motion with an instantaneous drift rate of 8%, and an instantaneous standard deviation of 20%. The initial market value of the bond is 63.32 and the risk-free rate is assumed to be 5%.

– 60.26). The required economic capital is 3.06 – the sum of the credit VAR estimate, $\text{VAR}^{V_0}_{\text{Credit}}(0.995, 1, 1) = -0.06$, plus funding debt interest, 3.12.

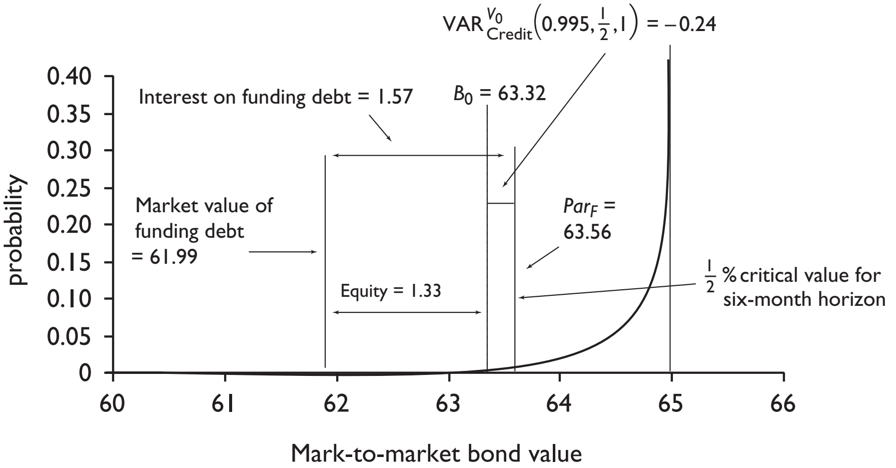
If, in this example, credit risk capital were set using the traditional credit VAR measure of unexpected credit losses – the difference between the mean of the end-of-period value distribution, 66.59 (not shown in Figure 3) and its 0.5% critical value, 63.38 – equity capital would be set equal to 3.21. In this example the traditional credit VAR approach overstates economic capital by 0.15, or about 5%.

5.3 Mark-to-market credit risk capital allocation

Suppose that the funding bond issued in the market risk example is purchased by another firm and funded for only six months. To estimate an unbiased capital allocation, the VAR horizon must be set equal to the maturity of the funding debt. Assume that the purchasing firm funds the issue with as much six-month debt as possible while limiting the default rate on its debt to 0.5% in this six-month interval. Figure 4 illustrates the capital allocation calculations in this example.

The end-of-period bond valuation distribution is generated using the BSM discount bond valuation equation (Expression (5)), in conjunction with the distribution for the purchased bond's supporting assets (Expression (29)) setting $T = ½$. Using this future asset value distribution, the 0.5% critical value of the BSM bond's end-of-period value distribution is 63.56, and its corresponding VAR measure, $\text{VAR}^{V_0}_{\text{Credit}}(0.995, ½, 1) = -0.24$, or *negative* 24 cents. The maximum par

FIGURE 4 Mark-to-market credit risk allocation.



Six-month mark-to-market credit risk capital allocation for a one-year BSM risky discount bond with a par value of 66.63 that is supported by assets that have an initial market value of 100 and future values that evolve according to geometric Brownian motion with an instantaneous drift rate of 8%, and an instantaneous standard deviation of 20%. The initial market value of the bond is 63.32 and the risk-free rate is assumed to be 5%.

value of the funding debt that can be issued without violating the target default rate constraint is 63.56. Expressions (26) and (30) are used to calculate the initial equilibrium value of the firm's funding debt which is 61.99. The equilibrium interest payment required by funding debt investors is 63.56 – 61.99 = 1.57. The required amount of equity funding needed to achieve the firm's funding objective is 1.33, the sum of $\mathrm{VAR}^{V_0}_{\mathrm{Credit}}(0.995, ½, 1) = -0.24$ and the required interest on funding debt, 1.57.

If capital were set using traditional credit VAR techniques, economic capital would be estimated to be 3.06, the 99.5% unexpected credit loss on this bond. In this case, credit VAR techniques overstate economic capital by 1.73, or by about 130%.

6 Conclusions

Buffer stock economic capital allocations cannot be accurately estimated using the VAR measures that are typically proposed in the literature. In both the market and credit risk setting, accurate capital allocation requires that VAR estimates be calculated relative to the initial market value of the assets or portfolio in question, and augmented with an estimate of the equilibrium interest cost on funding debt. Unlike in a short-horizon risk monitoring application, capital allocation VAR measures must accurately account for the expected return (or expected drift rate) that determines the future value of the assets (or portfolio). The interest compensation calculation and, in the case of non-traded or thinly traded debt instruments, an estimate of the initial mark-to-market value of the portfolio, will generally require the use of an asset pricing model.

While this paper discusses capital allocation using VAR techniques, the analysis also clearly demonstrates that there is really no need to calculate a VAR measure in order to calculate economic capital requirements. All that is actually required for this calculation is the critical value of the asset's end-of-period value distribution to set the par value of the funding debt, and an asset pricing relationship to estimate the equilibrium initial market value of the funding debt. The initial market value of the asset (portfolio) and the proceeds from the funding debt issue determine the required amount of equity needed to fund the asset (portfolio) purchase within the target default rate management criterion. This is true for both market risk and credit risk capital allocations. In the case of credit risk, the portfolio's so-called "unexpected loss" measure is irrelevant for constructing an economic capital allocation.

REFERENCES

Basel Committee on Banking Supervision (1995). *An Internal Model-Based Approach to Market Risk Capital Requirements*, Basel Switzerland.

Basel Committee on Banking Supervision (1999). "Credit risk modeling: Current practices and applications. Bank for International Settlements, Basel Switzerland (April).

Beder, Tanya (1995). VAR: Seductive but dangerous. *Financial Analysts Journal*, September/October.

Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy* **81** (May–June), 637–54.

CreditMetrics Technical Document (1997). New York: JP Morgan & Co.

Duffie, D. and Pan, J. (1997). An overview of value at risk, *Journal of Derivatives* **4**(3), 7–49.

Geske, R. (1977). The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis*, 541–52.

Geske, R. (1979). The valuation of compound options. *Journal of Financial Economics* **7**, 63–81.

Hull, J. and White, A. (1998). Value at risk when daily changes in market variables are not normally distributed. *Journal of Derivatives* **5**(3), 9–19.

Jorion, Philippe (1995). Measuring the risk in value at risk. *Financial Analysts Journal*, November/December.

Jorion, Philippe (1997). *Value at Risk*. New York: McGraw Hill.

Kupiec, P. (1999). Risk capital and VAR. *Journal of Derivatives* **7**(2), 41–52.

Kupiec, P. (2001a), Estimating credit risk capital: What's the use? *The Journal of Risk Finance* **2**(3), 17–34.

Kupiec, P. (2001b), What exactly does credit VAR measure? *The Journal of Derivatives* **9**(3), 46–59.

Marshall, C. and Siegel, M. (1997). Value-at-Risk: Implementing a risk measurement standard. *Journal of Derivatives* **4**(3), 91–110.

Matten, C. (1997). *Managing Bank Capital: Capital Allocation and Performance Measurement*. Chichester, England: John Wiley & Sons.

Merton, Robert (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance* **29** (May), 449–70.

JP Morgan (1997). CreditMetrics Technical Document. New York: JP Morgan & Co., Inc.

JP Morgan/Reuter (1996). RiskMetrics Technical Document, Fourth edition. New York: Morgan Guarantee Trust Company of New York.

Saunders, T. (1999). *Credit Risk Measurement*. New York: John Wiley & Sons.

Smithson, C. (1997). Capital budgeting. *Risk* **10**(6), 40–1.

Wilson, T. (1997). Portfolio credit risk (I). *Risk* **10**(9), 111–7.

Wilson, Thomas (1997). Calculating risk capital. In *The Handbook of Risk Management and Analysis*, ed. Carol Alexander. Chichester, England: John Wiley & Sons, pp. 193–232.