Joining the SABR and Libor models together

Fabio Mercurio and Massimo Morini propose a Libor market model consistent with SABR dynamics and develop approximations that allow for the use of the SABR formula with modified inputs. They verify that the approximations are acceptably precise, imply good fitting of market data and produce regular Libor rate parameters. They finally show that the correct assessment of the no-arbitrage volatility drift leads to a more sensible pricing of derivatives not included in the calibration set

The SABR model is a stochastic volatility dynamics for a single asset under its natural probability measure. However, when pricing general term structure payouts, we need to model the joint evolution of relevant rates, as in the Libor market model (LMM) of Brace, Gatarek & Musiela (1997). Moreover, if the stochastic volatility factors are correlated with the term structure of rates (as in Hagan *et al*, 2002, the underlying is correlated to its stochastic volatility), no-arbitrage constraints indicate that the stochastic differential equation governing the volatility evolution will be different under different swap and forward measures. This implies that when one assumes a factor structure for the stochastic volatility, as is usually done in the market for computational reasons, even the pricing of simple derivatives such as caps and swaptions is not trivial, because different volatility dynamics must be considered for different underlying rates.

Assuming non-zero correlation between rates and their volatility has several advantages, which overcome the cost of a volatility drift that is measure-dependent. The quality of calibration improves and model parameters tend to be more stable over time. Finally, an incorrect (that is, equal to zero) specification of the correlation can easily lead to undesired distributions of the final profit and loss of the hedging portfolio, resulting in unpredicted (and of course unwanted) potential losses.

In the following, we build a general LMM starting from SABR assumptions, and we show how the arbitrage-free dynamics can be approximated to reach a model where a no-arbitrage change of dynamics is captured by a simple modification in the parameters of the SABR formula. Then these approximations are tested under numerical methods. Finally we show that the prices calculated with our model, whose pricing formulas take into account no-arbitrage constraints, are closer to quoted market prices than the prices calculated with a model neglecting the issue. This confirms that compliance with no-arbitrage constraints is an important practical issue in the interest rate derivatives market. Due to market liquidity and the availability of information on volatility and correlations, the calculation of precise no-arbitrage corrections is possible and its importance is magnified by the large notional amounts.

The issue of reconciling the SABR dynamics with the LMM setting was first addressed by Henry-Labordère (2007) and then by Rebonato (2007). However, these two important contributions follow routes different from ours. Henry-Labordère derives a new approximation for swaption implied volatilities that involves matrix inversions and partial derivatives to calculate numerically, while we devise a formula that is very simply implemented through a change of parameters in the popular Hagan *et al* (2002) formula.

In terms of compatibility with the popular SABR formula, our work is more similar to Rebonato (2007). However, Rebonato does not design a LMM starting from the reference SABR dynamics, but adapts his LMM so as to obtain results as close as possible to those of the SABR formula. He performs an in-depth analysis of the most convenient parameterisation allowing a stable (time-homogeneous) fit to market data, and extends part of its analysis to a plurality of stochastic volatility factors. On the other hand, he does not consider the issue of the change of dynamics of the stochastic volatility required to keep the model arbitrage-free; that instead is the focus of our work.

Recently, the issue of reconciling SABR dynamics with the LMM setting has also been addressed by Hagan & Lesniewski (2008), who use small noise techniques.

The model formulation

Most stochastic volatility LMMs introduced in the financial literature allow for only one stochastic volatility factor applied to all forward rates. This includes the already mentioned Henry-Labordère (2007) and Wu & Zhang (2006), and also Piterbarg (2005) and Andersen & Andreasen (2002). Accordingly, in this work, we consider a single volatility factor. We also assume a single constant elasticity of variance parameter β, namely a common exponent in the diffusion coefficient of all the modelled forward rates. This does not harm our chance to fit precisely the different skews of different swaptions, since we have correlations with volatility that are different for each rate. On the other hand, this specification allows a regularity of parameters that would be unreachable when using different volatility backbones and leads to (approximated) swap rate dynamics that are easier to define.

In general, a parsimonious model design in terms of parameters and volatility factors avoids over-parameterisation and instability over time. A large number of factors leads to computationally burdensome models when using Monte Carlo. Even more importantly, the relevant common factors driving the market could be missed when each rate is calibrated independently of the others. For example, for many exotic derivatives, such as constant maturity swaps, out-of-sample model implications are more relevant than extremely good fitting, so that a parsimonious model can provide more meaningful prices (see Mercurio & Pallavicini, 2006, and Mercurio & Morini, 2007b).

In light of the above considerations, we propose a synthetic SABR LMM, whose formulation coincides with that introduced by Henry-Labordère (2007). The model dynamics is defined under the standard reference measure for stochastic volatility Libor models, namely the spot Libor measure Q, associated with the discretely rebalanced bank account numeraire $B_d(t)$. The bank account starts at one and is rebalanced only at the times that appear in the LMM discrete tenor structure. In the following:

$$F_j(t) := \frac{P(t,T_{j-1}) - P(t,T_j)}{\tau_j P(t,T_j)}$$

is the forward Libor rate at time t for the future interval $[T_{j-1}, T_j]$, τ_j is the year fraction from T_{j-1} to T_j and $P(t, T)$ denotes the time t-price of the zero-coupon bond with maturity T.

As in Henry-Labordère (2007), we assume that under Q the state variables follow SABR-like dynamics:

$$\begin{aligned} dF_k(t) &= \mu_k^Q(t)dt + \sigma_k V(t) F_k(t)^\beta dZ_k^Q(t) \\ dV(t) &= \upsilon V(t) dW^Q(t), \qquad V(0) = \alpha \end{aligned} \tag{1}$$

where $\mu_k^Q(t)$ is the spot Libor drift of forward rates, σ_k is a deterministic (constant) instantaneous volatility coefficient, $\beta \in (0, 1)$ is the constant elasticity of variance (local-volatility) coefficient and $Z^Q := \{Z_1^Q, Z_2^Q, ...\}$ and W^Q are Q-Brownian motions, whose instantaneous correlation structure is given by:

$$\mathbb{E}\left[dZ^Q(t)dZ^Q(t)'\right] = \rho dt, \quad \mathbb{E}\left[dW^Q(t)dZ_k^Q(t)\right] = \rho_{k,V}dt \quad \forall k$$

where ρ is some correlation matrix, and the vector $\rho^V = [\rho_{1,V}, \ldots, \rho_{M,V}]'$ expresses the rate-volatility instantaneous correlation. With no loss of generality, hereafter we will assume $\alpha = 1$ since α can be incorporated in σ_k.

Dynamics under different measures

Applying classic change-of-measure techniques, we can easily calculate joint no-arbitrage dynamics under other typical pricing measures (see Mercurio & Morini, 2007b, for detailed derivations). Under Q^i, the T_i-forward measure with numeraire $P(t, T_i)$, the dynamics of the forward rate $F_k(t)$ is in line with standard LMM results. For instance, if $i < k$:

$$\begin{aligned} dF_k(t) = {} & \sigma_k V^2(t) F_k(t)^\beta \sum_{j=i+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j F_j(t)^\beta}{1+\tau_j F_j(t)} dt \\ & + \sigma_k V(t) F_k(t)^\beta dZ_k^i(t) \end{aligned}$$

On the contrary, the result for the volatility process V is not standard. In fact, the Q^i-dynamics of V is:

$$\begin{aligned} dV(t) &= -\upsilon V^2(t)\mu_t(\gamma(t),i)dt + \upsilon V(t)dW^i(t) \\ \mu_t(x,y) &:= \sum_{j=x+1}^{y} \frac{\tau_j \rho_{j,V} \sigma_j F_j^\beta(t)}{1+\tau_j F_j(t)} \end{aligned} \tag{2}$$

(see also the change of measure performed in Henry-Labordère, 2007). A similar result applies when we consider a swap rate $S_{a,b}(t) = \Sigma_{j=a+1}^{b} w_j(t) F_j(t)$, where:

$$w_j(t) := \tau_j P(t,T_j) / C_{a,b}(t), \quad C_{a,b}(t) := \sum_{h=a+1}^{b} \tau_h P(t,T_h)$$

and move to the corresponding swap measure $Q^{a,b}$, whose associated numeraire is the swap annuity $C_{a,b}(t)$. Under $Q^{a,b}$, the dynamics of V is:

$$\begin{aligned} dV(t) &= -\upsilon V^2(t)\mu_t^{a,b}(\gamma(t))dt + \upsilon V(t)dW^{a,b}(t) \\ \mu_t^{a,b}(\gamma(t)) &= \sum_{k=a+1}^{b} w_k(t)\mu_t(\gamma(t),k) \end{aligned} \tag{3}$$

Applying the rather standard techniques of Andersen & Andreasen (2002) and Jäckel & Rebonato (2000), and setting:

$$\begin{aligned} \sigma_{a,b} &:= \sqrt{\sum_{k=a+1}^{b}\sum_{h=a+1}^{b} \gamma_k(0)\sigma_k\gamma_h(0)\sigma_h\rho_{k,h}}, \\ \gamma_j(t) &:= \frac{\partial S_{a,b}(t)}{\partial F_j(t)} \frac{F_j^\beta(t)}{S_{a,b}^\beta(t)} \end{aligned} \tag{4}$$

we obtain the (approximated) stochastic differential equation followed by the swap rate:

$$dS_{a,b}(t) = \sigma_{a,b} V(t) S_{a,b}^\beta(t) dZ_{a,b}^{a,b}(t)$$

whose instantaneous correlation $\rho_{a,b}^V$ with the stochastic volatility factor is given by:

$$\rho_{a,b}^V \sigma_{a,b} = \sum_{j=a+1}^{b} \gamma_j(0)\sigma_j\rho_{j,V} \tag{5}$$

We refer to these passages as the LMM-SMM (swap market model) approximation, which will be tested below. The involved drift terms in the volatility dynamics (2) and (3) prevent us from having closed-form formulas for caps and swaptions, which are essential requirements for efficient model calibrations. In the following section, therefore, we will approximate them so as to reach the required tractability. Precisely, we will propose different approximations, which will be tested empirically in a later section. Given the fact that $\mu_t(\gamma(t), k) = \mu_t^{k-1,k}(\gamma(t))$, since a forward Libor rate is equivalent to a one-period forward swap rate, we will concentrate on the volatility dynamics (3) under the swap measure $Q^{a,b}$.

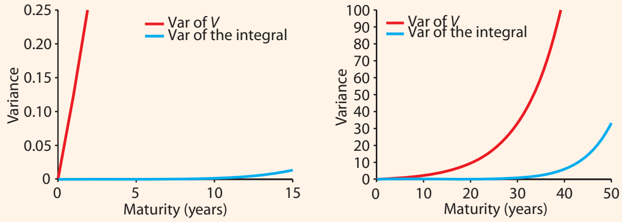
Tractable approximations of volatility dynamics

The first approximation for tractability one can think of is the trivial one where the drifts are set to zero:

$$\mu_t^{a,b}(\gamma(t)) \approx 0 \tag{6}$$

This corresponds to the roughest possible 'projection' of the true dynamics on to a lognormal dynamics consistent with the SABR formula. We refer to it as the trivial lognormal approximation. When correlations are taken to realistic, calibrated values, as we do

1. Comparing the variance of $\int_0^t c(u)\tilde{V}(u)du$ with that of $\tilde{V}(t)$



Note: the two graphs represent the same function on two very different scales

here, the trivial projection (6) can lead to considerable out-of-sample errors (see also the section below on empirical tests).

In this section, we look for a less rough approximation. Wu & Zhang (2006), consistently with all the literature on the LMM, observe that the variability of a forward-measure drift correction is negligible compared with the variability of the underlying state variables, so it is safe to approximate the values $F_j(t)$ that appear in the drift correction with their initial value $F_j(0)$. When it comes to swap measures, as shown by Rebonato & Jäckel (1999), a similar result holds thanks to the low variability of the weights $w(t)$ in $\mu_t^{a,b}(\gamma(t))$. In our context, this leads to:

$$\mu_t^{a,b}\left(\gamma(t)\right) \approx \mu_0^{a,b}\left(\gamma(t)\right) \tag{7}$$

One may argue that $\mu_t^{a,b}(\gamma(t))$ tends to be more volatile than the drifts obtained by Wu & Zhang (2006), since they, among other differences, assume $\beta = 1$. However, we have to notice that β is concurrent to $\rho_{j,V}$ in fitting the skewness in market volatility quotes. In case it happened that $\beta \ll 1$, this is typically associated with values $\rho_{j,V}$ very close to zero, setting both $\mu_t^{a,b}(\gamma(t))$ and its volatility to extremely low values.

Projecting volatility on to lognormal dynamics. Contrary to Wu & Zhang (2006), who, after a measure change, find dynamics analogous to that they started from in their reference measure, the approximation (7) leads to a dynamics, namely:

$$dV(t) = -\upsilon V^2(t)\mu_0^{a,b}\left(\gamma(t)\right)dt + \upsilon V(t)dW^{a,b}(t) \tag{8}$$

which is no longer driftless as in (1), nor of the geometric Brownian motion type we started from. This can cause two problems. First, this new solution could have a less regular behaviour. Second, it could prevent us from using the SABR option pricing formula, the feature of the SABR model that is responsible for its popularity in the market. We would like to incorporate this no-arbitrage dynamics into the same SABR formula, with just a correction to the input parameters. This means first of all performing what we call lognormal approximation (LA), namely projecting the above dynamics into some lognormal dynamics, as done in the trivial projection (6), but now with a behaviour as similar as possible to the true dynamics.

The first (still rough) improvement on (6) is to freeze to their time-zero value all stochastic quantities in the change-of-measure correction, including stochastic volatility $V(t)$ frozen to its initial level $V(0) = 1$. This leads to the following:

$$\text{Equation-based LA}: dV(t) = M(t)V(t)dt + \upsilon V(t)dW^{a,b}(t)$$
$$M(t) = M_1(t) := -\upsilon\mu_0^{a,b}\left(\gamma(t)\right) \tag{9}$$

Another approximation we want to investigate is obtained by directly tackling the stochastic differential equation (8). Solving the equation would allow us to assess the behaviour of (8) and help us in developing a better-founded approximation. Setting $c_t := -\upsilon\mu_0^{a,b}(\gamma(t))$, equation (8) can be written as:

$$dV(t) = c_t V(t)^2\,dt + \upsilon V(t)dW^{a,b}(t)$$

Mercurio & Morini (2007b) show that this stochastic differential equation admits an explicit solution given by:

$$V(t) = \tilde{V}(t)\varphi_t, \qquad \varphi_t = \left[1 - \int_0^t c_u \tilde{V}(u)du\right]^{-1} \tag{10}$$

where the auxiliary process $\tilde{V}(t)$ follows the standard SABR dynamics[1]:

$$d\tilde{V}(t) = \upsilon\tilde{V}(t)dW^k(t); \qquad \tilde{V}(0) = 1$$

The term φ_t represents the correction made to the stochastic volatility to take into account no-arbitrage constraints. Thus, a correct account of no-arbitrage leads to a 'convexity adjustment' that depends on some average value of the stochastic volatility from now to the option maturity. This 'convexity correction' φ_t is a stochastic quantity, so $V(t)$ is not lognormal. To use the SABR formula, we should first find a valid lognormal approximation, which requires approximating the convexity adjustment with a deterministic function. As also done before, this will be accomplished by replacing some not-so-volatile quantity with its expected value. To this end, we must find a quantity giving a reduced contribution to the total variability of $V(t)$. A good candidate is the integral $\int_0^t c_u\tilde{V}(u)du$, considering that the average of a stochastic process tends to be less volatile than the process itself. Mercurio & Morini (2007b) show this is indeed the case, by calculating the first and second moments of the integral $\int_0^t c_u\tilde{V}(u)du$. They find that the variance of the integral is negligible compared with its expected value, and also compared with the variance of $\tilde{V}(t)$, which is the dominant stochastic term in (10).

In figure 1, we show, using parameters from calibration to the swaption market, that the volatility of $\int_0^t c_u\tilde{V}(u)du$ remains very low compared with the volatility of $\tilde{V}(t)$, even for very long maturities. Accordingly, $\int_0^t c_u\tilde{V}(u)du$ can be approximated, with a good degree of accuracy, with its expected value:

$$V(t) \approx \frac{\tilde{V}(t)}{1 - \mathbb{E}^{a,b}\left[\int_0^t c(u)\tilde{V}(u)du\right]} = \frac{\tilde{V}(t)}{1 - \int_0^t c(u)du}$$

This corresponds to a volatility process given by the following:

$$\text{Solution-based LA}: dV(t) = M(t)V(t)dt + \upsilon V(t)dW^{a,b}(t)$$
$$M(t) = M_2(t) := \frac{c(t)}{1 - \int_0^t c(u)du} \tag{11}$$

Using the SABR option pricing formula. Both the equation-based LA and the solution-based LA dynamics:

[1] *Notice that, in principle, this solution can have explosive behaviour, depending on the sign and the magnitude of the piece-wise constant function c_t. However, we point out that (10) applies to the forward measure Q^k, whose numeraire expires at T_k, which is usually the maturity of the payout one needs to evaluate. With market values of model parameters, c_t can actually be positive but it is extremely low, so that the expected explosion time is beyond any possible financial maturity. This will be confirmed below by numerical tests*

$$dV(t) = M(t)V(t)dt + \upsilon V(t)dW^{a,b}(t) \qquad (12)$$

share the same kind of (lognormal) distribution as the standard, single-asset SABR model. However, the popular SABR formula comes from a driftless volatility with constant parameters, whereas here we have a volatility drift $M(t)$, which is even time-dependent. Integrating (12), the volatility $V(T_a)$ at maturity T_a is:

$$V(T_a) = e^{\int_0^{T_a} M(s)ds} e^{-\frac{1}{2}\upsilon^2 T_a + \upsilon W(T_a)} \qquad (13)$$

The same value at maturity T_a can be obtained by using driftless volatility dynamics and setting $V(0) = \exp\{\int_0^{T_a} M(s)ds\}$. With this modified initial value, we can recover the desired distribution at maturity T_a. However, European-style options do not depend only on the final value of volatility, but rather on the entire path followed by the volatility from zero to T_a. Thus, a more sensible approximation comes from matching the expected average volatility from the current time until maturity:

$$V(0) \approx \bar{V}_0^{a,b} := \frac{\mathbb{E}^{a,b}\left[\int_0^{T_a} V(t)dt\right]}{T_a} = \frac{\int_0^{T_a} e^{\int_0^t M(s)ds} dt}{T_a}$$

leading to:

$$\text{SABR-compatible LA}: dV(t) = \upsilon V(t)dW^{a,b}(t), \quad V(0) = \bar{V}_0^{a,b} \qquad (14)$$

where $\bar{V}_0^{a,b}$ is clearly measure-dependent. This is the final step to define our closed-form formula for swaption pricing. Using the Hagan *et al* (2002) notation, the SABR implied volatility $Vol(K, T)$ for an option with strike K and expiry T is given by a closed-form formula:

$$Vol(K,T) = SABR(\alpha, \beta, \rho, \nu, F_0, K, T)$$

In our LMM, the implied volatility $Vol_{a,b}(K, T_a)$ of a swaption with expiry T_a and tenor $T_b - T_a$ is given by:

$$Vol_{a,b}(K, T_a) = SABR\left(\bar{V}_0^{a,b}\sigma_{a,b}, \beta, \rho^V_{a,b}, \nu, S_{a,b}(0), K, T_a\right) \qquad (15)$$

One can devise more refined ways to map (12) into driftless dynamics. For example, one could calculate option prices with no further approximation by using the SABR formula with time-dependent parameters, also derived by Hagan *et al* (2002). However, this simple approximation based on the classic SABR formula already proves to be acceptably precise, as the following tests will show.

Testing the SABR LMM: numerical tests

We use antithetical variates and increase the number of scenarios so as to reduce the 98% one-sided Monte Carlo window on volatility to be less than 10 basis points. This is lower than the bid-ask spread on the most liquid swaptions (close-to-the-money, with maturities and tenors of a few years), often estimated at 30bp by expert traders. It is also much lower than the historical average swaption bid-ask considered in the literature (see Fan, Gupta & Ritchken, 2007, who estimate it to be around 50bp of volatility, and even higher for contracts less liquid because of large maturity/far-from-the-money strikes). For dealing with the practical possibility of negative values (theoretically not possible), we use a modification of the Higham & Mao (2005) algorithm.[2]

A. Swaption 5y5y: testing the formula versus Monte Carlo, with implied volatility errors

Strikes	Vol (MC)	Vol (formula)	Error (%)	Error (bp)
$S_{a,b}(0)$–1.00%	15.59%	15.77%	0.18%	–18
$S_{a,b}(0)$–0.50%	14.63%	14.72%	0.09%	–9
$S_{a,b}(0)$–0.25%	14.28%	14.34%	0.05%	–5
$S_{a,b}(0)$+0.00%	14.02%	14.04%	0.02%	–2
$S_{a,b}(0)$+0.25%	13.83%	13.83%	0.00%	0
$S_{a,b}(0)$+0.50%	13.71%	13.69%	–0.02%	2
$S_{a,b}(0)$+1.00%	13.62%	13.61%	–0.01%	1
$S_{a,b}(0)$+2.00%	13.92%	13.93%	0.01%	–1

B. Swaption 10y10y: testing the formula versus Monte Carlo, with implied volatility errors

Strikes	Vol (MC)	Vol (formula)	Error (%)	Error (bp)
$S_{a,b}(0)$–1.00%	12.87%	13.44%	0.57%	57
$S_{a,b}(0)$–0.50%	11.89%	12.19%	0.30%	30
$S_{a,b}(0)$–0.25%	11.51%	11.70%	0.19%	19
$S_{a,b}(0)$+0.00%	11.22%	11.32%	0.10%	10
$S_{a,b}(0)$+0.25%	11.00%	11.04%	0.04%	4
$S_{a,b}(0)$+0.50%	10.87%	10.87%	0.00%	0
$S_{a,b}(0)$+1.00%	10.82%	10.80%	–0.01%	–1
$S_{a,b}(0)$+2.00%	11.27%	11.35%	0.09%	9

The total errors between our closed-form swaption volatility formula (15) and the volatility obtained by the non-approximated Libor model with constant elasticity of variance rates and volatility dynamics (2) are reported, for a five-year/five-year swaption, in table A.

In spite of the approximations involved, the formula (15) appears definitely precise, with almost negligible errors for at-the-money and out-of-the-money swaptions, and an error by far within 30bp even in the worst case.

In this case, it is clearly difficult even to make a breakdown of the error. This task is more interesting when moving to the most challenging case in our dataset: the 10-year/10-year swaption. This is the swaption with the longest maturity and tenor among the liquid swaptions, which are those in the popular 10 × 10 × strikes swaption cube. On this option we show a breakdown of the total error into its different components. The total error is as shown in table B.

The formula still appears precise, with an error within the 30bp threshold for all swaptions but one. However, the error still appears acceptable even in the worst case, around 50bp, the minimum bid-ask considered in the related literature.

It is natural to wonder how this total error can be attributed to the different approximations involved. We attempt below the breakdown by comparing swaption volatilities calculated with the following dynamics:

i) LMM dynamics with:

$$dV_1(t) = -\upsilon V_1^2(t)\mu_t(\gamma(t), k)dt + \upsilon V_1(t)dW^k(t)$$

ii) LMM dynamics with:

[2] *Notice we do not assume an absorbing boundary at zero because in our tests it is not required for having a unique solution (we have β > 1/2), and because our rates have a no-arbitrage drift*

C. Testing the different approximations (implied vol errors in bp)

Strikes	Error $V_1 - V_2$ (lognormal approx)	Error $V_2 - V_3$ (SABR-compatible approx)	Error $V_3 - V_4$ (SMM-LMM approx)
$S_{a,b}(0)$–1.00%	0.5	3.8	–6.5
$S_{a,b}(0)$–0.50%	0.5	3.4	–1.1
$S_{a,b}(0)$–0.25%	0.5	3.3	1.0
$S_{a,b}(0)$+0.00%	0.5	3.2	2.6
$S_{a,b}(0)$+0.25%	0.6	3.2	3.4
$S_{a,b}(0)$+0.50%	0.8	3.4	3.9
$S_{a,b}(0)$+1.00%	1.1	3.9	4.5
$S_{a,b}(0)$+2.00%	1.9	5.1	4.4

D. Calibrated volatilities

k	1	2	3	4	5	6	7	8	9	10
σ_k	0.089	0.088	0.088	0.085	0.086	0.080	0.082	0.079	0.081	0.069
k	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	
σ_k	0.068	0.067	0.072	0.067	0.066	0.062	0.056	0.056	0.054	

$$dV_2(t) = V_2(t)M_2(t)dt + \upsilon V_2(t)dW^k(t)$$

iii) LMM dynamics with:

$$dV_3(t) = \upsilon V_3(t)dW^k(t), \qquad V_3(0) = \bar{V}_0^{k-1,k}$$

iv) SMM with swap rate instantaneous volatility $\sigma_{a,b}$, instantaneous correlation $\rho^V_{a,b}$, and stochastic volatility dynamics:

$$dV_4(t) = \upsilon V_4(t)dW^{a,b}(t), \qquad V_4(0) = \bar{V}_0^{a,b}$$

Since i) is the actual model dynamics (2) before any approximation and ii) is the solution-based dynamics (11) after the lognormal projection, the difference between i) and ii) is a good indication of the error due to what we called LA (which includes also the effect of drift freezing in (7)). The difference between ii) and iii) is an indication of the error due to the approximation of the lognormal dynamics $dV_2(t)$ with $dV_3(t)$, a driftless dynamics with modified initial volatility introduced to be compatible with the SABR formula (SABR-compatible approximation).

The difference between iii) and iv), instead, is the error due to the passage from an LMM to an SMM using approximations (4) and (5) (we called it the LMM-SMM approximation). Since in iv) the dynamics specification is the same underlying the closed-form formula (15), it is clear that the remaining error is due to the approximation error intrinsic in the Hagan *et al* (2002) SABR formula (which is well known to lose accuracy for long maturities). The results are reported in table C.

The results show that the lognormal approximation is well founded, leading to a very limited error.[3] The error due to the drift elimination in the SABR-compatible approximation (which could be avoided by moving to the SABR formula with time-dependent parameters) is higher but still limited. The error due to the passage from the Libor model to swap model parameters is significant, but still in line with results for similar approximation reported by Andersen & Andreasen (2002) and Jäckel & Rebonato (2000).

The major part of the error, as expected, is due to the Hagan *et al* (2002) approximation, and therefore unavoidable in any attempt to close, as much as possible, the gap between the market standard for simple interest rates derivatives (which uses the Hagan *et al* approximation) and the market standard for complex derivatives (which uses Libor models and would like them to be distributionally as close as possible to the SABR model). In particular, notice that it is the Hagan *et al* approximation that is responsible for the uneven distribution of the errors across strikes.

In the end, we introduce a market swaption with a very large maturity, the 30-year/five-year. In this case the formula may be expected to perform badly, since the precision of the Hagan *et al* (2002) approximation and the LMM-SMM approximation tends to worsen with the increase in variance of the underlying and of the stochastic volatility due to the increased maturity. Instead, the error due to the Hagan *et al* approximation and to the LMM-SMM approximation remains lower than 25bp in the worst case. Looking at the results of the test, this appears due to the fact that, when including long maturity swaptions, we include rates with lower volatility and smiles where the skew part is dominant compared with the convexity. This implies different rate-volatility correlations, and a different volatility of volatility, with some benefits for the performance of the two approximations tested.

After this numerical verification of our results, we can test their financial significance, in terms of regularity of the calibrated parameters and goodness of out-of-sample pricing.

Testing the SABR LMM: empirical tests

We test the model on euro swaption data as of June 8, 2006, considering liquid swaption smiles for maturities of one year, five years and 10 years, and tenors of two years, five years and 10 years, and also all at-the-money quotations of a standard 10 × 10 swaption matrix. We first test the case of zero correlations $\rho_{k,V}$ between interest rates and their volatility. In this case, the volatility drift is zero under any measure, and no approximation of the volatility dynamics is required for an explicit pricing of swaptions. We obtain a mean (absolute) calibration error of 23.24bp.

Assuming a general correlation $\rho_{k,V} \neq 0$, the volatility dynamics changes with the pricing measure.[4] There are two main issues we want to address: regularity of model parameters, possibly affected by the approximations used in the volatility drift; and implications for the implied prices of derivatives out of the calibration sample.

Using the solution-based dynamics (11), the calibration to the swaption data above can be performed with a mean error of 9bp, a clear improvement compared with that obtained under $\rho_{k,V} = 0$. As for the resulting parameters, the calibrated volatilities σ_k are regular and consistent with market patterns (see table D).

The volatility corrections $\bar{V}_0^{a,b}$ are quite stable and relatively small, which may give rise to suspicion that the no-arbitrage drifts on the volatility are irrelevant, and that equivalent results could already be

[3] With respect to the theoretically possible explosion of the volatility solution, we point out that, defining a 'volatility explosion' as an increase of volatility by 10 times compared with its initial level, in Monte Carlo tests for a 10-year/10-year swaption we find 1,510 'volatility explosions' every 100,000 scenarios when we just simulate a standard lognormal martingale volatility. When we simulate the true, potentially explosive dynamics, and the lognormal approximation that mimics its behaviour, we find, respectively, 1,589 and 1,599 'volatility explosions'. Being at the same level as a standard martingale (as confirmed also by the similarity of the resulting volatilities), the volatility explosion of the actual dynamics does not appear to be a practical problem for the maturities we see in the market (additionally, in the five-year/five-year simulation example, we had less than one 'explosion' every 1,000 scenarios)

[4] Notice that the global correlation matrix including cross-rate correlations and rate-volatility correlations must be positive semi-definite. We tackle this problem by using the correlation structure described in Mercurio & Morini (2007a)

found under the trivial approximation (6), which in fact gives a calibration error of 9.03bp with analogous regularity of parameters. However, the ability to fit market data with regular parameters is only one aspect to consider in judging a pricing formula. One of the most important practical aspects to consider is the effect on the pricing of derivatives not included in the calibration set. It is in such a test that we want to see if the equation-based (9) and solution-based dynamics (11) give results that are practically different from those obtained with the trivial 'no- drift' approximation (6), which radically neglects the no-arbitrage effects induced by a measure change. To this end, we limit the calibration set to smiles with maturities of one, two and 10 years, and we price, with the above approximations, swaptions not included in our calibration set. As an example, here we concentrate on the liquid five-year/two-year swaption.

The first finding is that the difference between the implied volatilities obtained with the trivial 'no-drift' approximation and the volatilities obtained with our equation-based and solution-based approximations appear higher than the bid-ask spread on such options, surpassing 100bp.[5] Secondly, we can compare the results obtained by the three approximations with the market quotations (see table E).

This simple case shows that the consistency with out-of-sample market quotes is clearly improved by taking into account the proper volatility drift correction through our approximations, in particular with the solution-based dynamics. In Mercurio & Morini (2007b), further tests on out-of-sample swaptions are reported.

E. Out-of-sample pricing

Strikes	Market vols	Error: trivial	Error: eq-based	Error: sol-based
$S_{a,b}(0)$–1.00%	16.86%	91.5	40.7	30.4
$S_{a,b}(0)$–0.50%	15.81%	47.2	17.5	12.4
$S_{a,b}(0)$–0.25%	15.45%	22.5	5.3	3.1
$S_{a,b}(0)$+0.00%	15.20%	3.7	7.5	6.5
$S_{a,b}(0)$+0.25%	15.04%	29.3	19.0	14.8
$S_{a,b}(0)$+0.50%	14.96%	53.6	28.9	21.7
$S_{a,b}(0)$+1.00%	15.01%	96.7	45.4	32.6
$S_{a,b}(0)$+2.00%	15.57%	150.8	61.0	40.7

Conclusions

We propose an arbitrage-free implementation of an LMM consistent with SABR dynamics, designed to remain simple and synthetic but providing, at the same time, a good fit to market data. We calculate the joint dynamics followed by Libor rates and stochastic volatility of the SABR kind under forward Libor and swap measures. The volatility's stochastic differential equation under a forward or swap measure turns out to be non-standard, compared with other results in the related literature. Based on the analysis of the equation found, we develop and justify theoretically approximations aimed at making these no-arbitrage adjustments compatible with the use of the SABR option pricing formula.

The approximations we develop are then tested numerically against Monte Carlo and empirically by calibration to real market data. We verify that the formulas properly accounting for no-arbitrage corrections are acceptably precise, allow for a good fitting of market data, and produce regular Libor parameters. When pricing derivatives out of the calibration sample, we find that our no-arbitrage drift corrections already have a non-negligible impact on vanilla options, and that the corrections make out-of-sample model prices closer to market quotes, compared with the prices implied by the trivial model, where such corrections are set to zero. ●

Fabio Mercurio is senior quant researcher at Bloomberg in New York. Massimo Morini is head of credit models and co-ordinator of model innovation at Banca IMI in Milan. **They are grateful to participants at the Interest Rate Derivatives week, WBS, March 2007, and the V Fixed Income Conference, 2007, for useful comments and discussion. This paper expresses the views of its authors and does not represent the opinion of Bloomberg or Banca IMI, and neither organisation is responsible for any use that may be made of its contents.**
Email: fmercurio@bloomberg.net, massimo.morini@bancaimi.com

[5] *This is also much higher than the difference between the true Monte Carlo price and the approximated formula for similar swaptions (see the five-year/five-year test in the previous section)*

REFERENCES

Andersen L and J Andreasen, 2002
Volatile volatilities
Risk December, pages 163–168

Brace A, D Gatarek and M Musiela, 1997
The market model of interest rate dynamics
Mathematical Finance 7, pages 127–154

Fan R, A Gupta and H Ritchken, 2007
On pricing and hedging in the swaption market
Journal of Derivatives, August, pages 9–33

Hagan P and D Lesniewski, 2008
Libor market model with SABR style stochastic volatility
Working paper

Hagan P, D Kumar, A Lesniewski and D Woodward, 2002
Managing smile risk
Wilmott Magazine, September, pages 84–108

Henry-Labordère P, 2007
Combining the SABR and LMM models
Risk October, pages 102–107

Higham D and X Mao, 2005
Convergence of the Monte Carlo simulations involving the meanreverting square root process
Journal of Computational Finance 8(3), pages 35–62

Jäckel P and R Rebonato, 2000
Linking caplet and swaption volatilities in a BGM/J framework: approximate solutions
Available at *www.quarchome.org/capletswaption.pdf*

Mercurio F and M Morini, 2007a
A note on correlation in stochastic volatility term structure models
Working paper, available at *SSRN.com*

Mercurio F and M Morini, 2007b
No-arbitrage dynamics for a tractable SABR term structure Libor model
Available at *SSRN.com*

Mercurio F and A Pallavicini, 2006
Smiling at convexity
Risk August, pages 64–69

Piterbarg V, 2005
Stochastic volatility model with time-dependent skew
Applied Mathematical Finance 12(2), June, pages 147–185

Rebonato R, 2007
A time homogeneous, SABR-consistent extension of the LMM
Risk November, pages 92–97

Wu L and F Zhang, 2006
Libor market model with stochastic volatility
Journal of Industrial and Management Optimization 2(2), May, pages 199–227