CVA pricing for commodities with WWR

This article focuses on calculating credit valuation adjustment (CVA) for commodity futures with wrong-way risk (WWR) and counterparty credit deterioration simultaneously. Kelin Pan presents an analytical expression that calculates CVA using integration of a commodity futures exposure and the conditional probability of a credit event under WWR and credit downgrades

The commodity futures price is one of the key market factors in valuing counterparty credit risk (CCR). Since a risky asset is priced by reducing the credit valuation adjustment (CVA) of a risk-free asset, the toughest aspect of CCR is CVA pricing. It increases the difficulty of pricing CVA when the correlation between counterparty exposure and a credit event – in this case, wrong-way risk (WWR) – is considered. According to Basel III (Bank for International Settlements 2010), the major credit losses during the credit crisis were due to counterparty credit downgrades or deterioration rather than actual default. Adopting this point of view, the CVA model should also incorporate credit deterioration.

Several research papers have studied CVA with WWR. Pykhtin & Rosen (2010) adopted a Gaussian copula model to calculate WWR and obtained an analytical expression using a normal credit exposure assumption. Hull & White (2012) postulated a linear model between exposure and default. Rosen & Saunders (2012) developed a simulation algorithm to capture the CVA for both general and specific WWR. Among commodities pricing models, Gabillon (1991) and Gibson-Schwartz (1990) showed two classical models for crude oil futures.

In this article, we present a CVA with WWR model for commodity futures contracts. The credit exposure for commodity futures is calculated based on the Gabillon (1991) two-factor model. The model parameters are calibrated to market data for crude oil futures. The credit deterioration indicator is calculated using the Standard & Poor's (S&P) credit transition matrices (Vazza & Kraemer 2013). An analytical expression for commodity CVA with WWR is obtained. The numerical results show that CVA is a function of market-credit correlation, maturity time, risk-rating grade and credit transition period.

Commodity pricing model

A popular model for commodity futures is the Gabillon (1991) model. The advantage of the Gabillon model is the analytical expression and easy estimation of model parameters it provides. Using the analogy of a cantilever can explain oil futures prices as well as commodity futures prices, provided the latter follow the cantilever assumption. The two state variables in the Gabillon model are the short- and long-term price of crude oil (S and L, respectively), which are two stochastic processes:

$$\begin{aligned} \mathrm{d}S &= \mu_{S,t}\,\mathrm{d}t + \sigma_{S,t}\,\mathrm{d}W_{t1} \\ \mathrm{d}L &= \mu_{L,t}\,\mathrm{d}t + \sigma_{L,t}\,\mathrm{d}W_{t2} \end{aligned}$$

where $\mathrm{d}W_{t1}$ and $\mathrm{d}W_{t2}$ are two correlated Wiener processes with $\mathrm{d}W_{t1}\,\mathrm{d}W_{t2} = \rho\,\mathrm{d}t$, and ρ is the correlation coefficient of the two Wiener processes. The short-term price is the nearest traded contract. For example, on March 19, 2018, the nearest contract was CLJ18, delivering in April 2018. The long-term price could be CLJ23, delivering in April 2023. The model on crude oil futures price is a function of L, S, t and T (maturity). Using Itô's lemma, the crude oil futures price function yields:

$$\mathrm{d}F = \mu^{\mathrm{F}}\,\mathrm{d}t + \sigma^{\mathrm{F}} F\,\mathrm{d}W_t$$

where μ^{F} and σ^{F} are the drift (Gabillon 1991, equation (13)) and the volatility (Gabillon 1991, equation (23)) of futures price, respectively. The risk-neutral property of futures price implies a drift term of zero for the above equation, which leads to the following solution (Gabillon 1991, equations (19) and (24)):

$$F = A_{t,T} S^{B_{t,T}} L^{1-B_{t,T}}$$

where:

$$\left.\begin{aligned} B_{t,T} &= \exp\left\{-\int_t^T \eta_u\,\mathrm{d}u\right\} \\ A_{t,T} &= \exp\left\{\int_t^T [(r-\delta_u)B_{u,T} + \tfrac{1}{2}\nu_u B_{u,T}(B_{u,T}-1)]\,\mathrm{d}u\right\} \end{aligned}\right\} \qquad (1)$$

In addition, r is the interest rate, while η_u, δ_u and ν_u are model parameters, calibrated using market data. The dynamic of futures price under the risk-neutral property is:

$$\mathrm{d}F_{t,T} = \sigma^{\mathrm{F}}_{t,T} F_{t,T}\,\mathrm{d}W_t \qquad (2)$$

$$\sigma^{\mathrm{F}}_{t,T} = [\sigma_{\mathrm{S}}^2 B_{t,T}^2 + \sigma_{\mathrm{L}}^2(1-B_{t,T})^2 + 2\rho\sigma_{\mathrm{S}}\sigma_{\mathrm{L}} B_{t,T}(1-B_{t,T})]^{1/2} \qquad (3)$$

where σ_{S} and σ_{L} are the volatilities of the short- and long-term price, respectively. Equation (2) implies the futures price is lognormally distributed. Using Itô's lemma for the function $f(F) = \ln F$ yields:

$$F_{t,T} = F_{0,T}\exp\left\{-\frac{1}{2}\int_0^t (\sigma^{\mathrm{F}}_{u,T})^2\,\mathrm{d}u + \int_0^t \sigma^{\mathrm{F}}_{u,T}\,\mathrm{d}W_u\right\}$$

This process was not discussed by Gabillon (1991). It was, however, discussed by Schwartz & Smith (2000, page 899, paragraph 1). In the latter paper, the Itô process is used for function e^X, where $X = \ln(F)$. We use function $\ln(F)$ directly. Therefore, the first term of the power function is negative. When $t = T$, we have $F_{T,T} = S_T$. Thus, the spot price at future time T is expressed as:

$$S_T = F_{0,T}\exp\left\{-\frac{1}{2}\int_0^T (\sigma^{\mathrm{F}}_{u,T})^2\,\mathrm{d}u + \int_0^T \sigma^{\mathrm{F}}_{u,T}\,\mathrm{d}W_u\right\} \qquad (4)$$

We further assume the volatility of the futures depends only on today's information (calculation date t). This implies $\sigma^{\mathrm{F}}_{u,T} = \sigma^{\mathrm{F}}_{t,T}$. Thus, (4) becomes:

$$S_T = F_{0,T}\exp\{-\tfrac{1}{2}(\sigma^{\mathrm{F}}_{t,T})^2 T + \sigma^{\mathrm{F}}_{t,T}\sqrt{T}Z\} \qquad (5)$$

A. Parameters of the commodity futures pricing model						
Parameter	**CLJ18**	**CLJ19**	**CLJ20**	**CLJ21**	**CLJ22**	**CLJ23**
Volatility	1.944	1.515	1.209	1.118	1.072	1.070
B_{0,T_j}	1.000	0.812	0.666	0.619	0.594	0.593
b_{T_j}	−0.1176	0.0693	0.0908	−0.0512	0.0092	−0.0006

Data source: footnote 1

where Z is a standard normal random variable. Note $F_{0,T}$ is the futures price at contract time $t = 0$. According to (3) and (5), three parameters σ_S, σ_L and $B_{0,T}$ are required to be calibrated by market data. σ_S and σ_L are calibrated using market data directly, eg, by employing CLJ18 and CLJ23 contracts, as mentioned previously. In order to estimate $B_{0,T}$ via (1), we assume η_t follows a five-order polynomial function instead of the constant employed by Gabillon (1991):

$$\eta_t = \sum_{i=0}^{5} a_i t^i$$

Substituting the above function into (1) yields:

$$B_{t,T} = \exp\left\{-\sum_{i=0}^{5} b_i (T^{i+1} - t^{i+1})\right\} \tag{6}$$

where T and t are maturity and current time, respectively, and $b_i = a_i/(i+1)$ $(i = 0, 1, \dots, 5)$. In order to estimate b_i, the following six time series are created using crude oil futures contract data from December 20, 2017 to March 19, 2018: CLJ18, CLJ19, CLJ20, CLJ21, CLJ22 and CLJ23 for April-traded contracts.[1]

In the United States, West Texas Intermediate (WTI) crude oil futures are traded on the Nymex under ticker symbol CL: CLF, CLG, CLH, CLJ, ..., CLZ, and delivered in January, February, March, April, ..., December. Each contract has a size of 1,000 barrels, and the price is in US dollars per barrel. The data provides two dates: the current time t and the delivery time T. For example, the market price on January 19, 2018 for CLJ19 is US$58.65, which implies a delivery date of April 2019. Since the shortest term is April (as of March), we consider CLJxx time series data to capture both short- and long-term futures contract information. For the available data, CLJ18 and CLJ23 are proxies of the short- and long-term prices, respectively. If we take the start time as $t = 0$, (6) is reduced to:

$$B_{0,T_j} = \exp\left\{-\sum_{i=0}^{5} b_i T_j^{i+1}\right\}, \quad j = 0, 1, \dots, 5 \tag{7}$$

where $T_0 = 0$ (2018), $T_1 = 1$ (2019), ..., $T_5 = 5$ (2023), eg, $T_3^{4+1} = 3^5$. The standard deviation of six data sets is given in row 1 of table A.

Table A reveals the volatility decreases as maturity increases. As assumed by Gabillon (1991), the volatility of the long-term price should tend to zero at the end of the cantilever. However, in reality, the observed long-term oil price has a finite maturity; therefore, it is a stochastic process but with a smaller volatility than that of the short-term price. In this case, using proxies of the short- and long-term prices is reasonable. Substituting $\sigma_S = \sigma_{CLJ18}$, $\sigma_L = \sigma_{CLJ23}$ and $\sigma^F_{0,T} = (\sigma_{CLJ18}, \dots, \sigma_{CLJ23})$ into (3), the parameter $B_{0,T}$ is solved by a square root equation and listed in row 2 of table A. Once the matrix $\vec{B}$ is found, the parameter matrix $\vec{b}$ is computed by solving:

$$-\ln \vec{B} = \boldsymbol{X} \cdot \vec{b}$$

where $x_{j,i} = T_j^{i+1}$ based on maturity T_j $(0, 1, \dots, 5)$ and power i $(0, 1, \dots, 5)$. The computed results are shown in row 3 of table A. $B_{t,T}$ can be extrapolated to any maturity time. Another parameter $A_{t,T}$ in (1) can be calibrated to the current futures contract price using estimated $B_{t,T}$. Since the CVA pricing model relies only on the futures volatility, which is a function of $B_{t,T}$, we will not discuss the calibration process of $A_{t,T}$ in this article.

The expected exposure for commodity futures is expressed under the risk-neutral measure Q (Pykhtin & Rosen 2010):

$$S_t = E^Q[P_{t,T} S_T] \tag{8}$$

where $P_{t,T}$ is the discount factor. In the next section, we will discuss the probability of a credit event and introduce the concept of a credit deterioration indicator.

Probability of a credit event

A credit event occurs once the CCR rating is downgraded. The credit event time τ_i can be mapped to a normal cumulative density function (CDF) with credit criterion $C_{i,t}$:

$$\mathrm{PD}_{i,t} \equiv \Pr(\tau_i \leqslant t) = \Phi(C_{i,t})$$

For a homogeneous portfolio, $C_{i,t} = C_t$, which is a deterministic function, and:

$$C_t = \Phi^{-1}(\mathrm{PD}_t) \tag{9}$$

where $\Phi^{-1}(\cdot)$ is the inverse normal CDF. Under a Gaussian copula model, the entity (counterparty) asset value is driven by a systematic common factor (Y) and an idiosyncratic factor (ε_t):

$$A_t = \beta Y + \sqrt{1-\beta^2}\,\varepsilon_t \tag{10}$$

where β is the correlation coefficient between asset value A_t and risk factor Y. We define Y as the credit deterioration indicator. Note β is negative and Y is positive (see table C), while A_t may be positive or negative depending on the idiosyncratic factor ε_t. When credit decreases, the asset value is reduced until $A_t < C_t$, which leads to a well-known formula for conditional probability of default (PD):

$$\begin{aligned}\mathrm{PD}(t \mid Y) &= \Pr(A_t \leqslant C_t \mid Y) \\ &= \Pr(\beta Y + \sqrt{1-\beta^2}\,\varepsilon_t \leqslant C_t) = \Phi\left(\frac{C_t - \beta y}{\sqrt{1-\beta^2}}\right)\end{aligned} \tag{11}$$

where C_t is given by (9). To estimate β, we simply assume the counterparty's asset is assessed by its stock price. As an example, β is obtained by calculating the Pearson correlation coefficient between the Citibank stock (as counterparty) and the S&P 500 Index (as system factor). The data is from October 13, 2015 to February 5, 2018.[2] The result is $\beta = -80.63\%$. The negative sign is due to the opposite direction of the S&P 500 Index and the credit deterioration indicator.

[1] *See www.barchart.com/futures/quotes/CL*0/all-futures.*

[2] *See https://finance.yahoo.com/quote.*

B. Rescaled transition matrices (%)				
	One-year		Three-year	
Rating	AAA	AA	AAA	AA
AAA	90.049	0.531	72.874	1.445
AA	9.270	89.575	24.134	72.906
A	0.392	9.279	1.848	22.612
BBB	0.082	0.469	0.374	2.303
BB	0.082	0.031	0.242	0.214
B	0.041	0.031	0.099	0.214
C	0.082	0.042	0.187	0.045
D	0.000	0.042	0.242	0.260

Data source: Vazza *et al* (2013, table 41)

C. Indicator Y for AAA and AA obligors				
	One-year		Three-year	
Rating	AAA	AA	AAA	AA
AAA	1.28	−2.55	0.61	−2.18
AA	2.47	1.29	1.88	0.65
A	2.76	2.50	2.28	1.88
BBB	2.87	2.98	2.42	2.44
BB	3.03	3.05	2.56	2.56
B	3.15	3.14	2.63	2.74
C	>3.15	3.34	2.82	2.79
D	>3.15	>3.34	>2.82	>2.79

Table B shows credit transition probabilities based on the S&P RatingsDirect Report (eg, Vazza & Kraemer 2013). Denoting the transition probability from state i to state j as $q_{i,j}$, the relationship between $q_{i,j}$ and the CDF of Y can be expressed as:

$$q_{i,j} = \Phi(Y_{i,j}) - \Phi(Y_{i,j-1}) \tag{12}$$

assuming $\Phi(Y_{i,0}) = 0$ and $i = 1,2$ (corresponding to AAA, AA), $j = 1,\dots,8$ (corresponding to AAA, ..., D). The credit deterioration indicator Y is solved via (12) using an iterative approach. The results are shown in table C.

A positive value of Y indicates a credit downgrade, while a negative value of Y indicates a credit upgrade. For CVA calculations, the upper limit only takes positive values, which correspond to a credit downgrade. The last state in table C is defined by D, which gives the lower boundary of credit events. For example, for AAA obligors with a one-year average transition probability, the last state is specified by $Y_s = 3.15$. For AA obligors, the last state is specified by $Y_s = 3.34$.

CVA WWR

WWR is the case when the exposure positively depends on the PD or a credit event. The exposure increases when the credit deterioration indicator is high. However, the correlation between the exposure and the credit event is hard to compute. It may vary with different market conditions and different assets. In this article, a one-factor Gaussian copula model is adopted to correlate the market factor with credit deterioration. Under the Gaussian copula model, the market factor X is positively correlated with the credit deterioration indicator Y as follows:

$$X = \rho Y + \sqrt{1-\rho^2}\,\omega \tag{13}$$

where ρ is the correlation coefficient between the market factor and the credit deterioration indicator. Equation (13) captures the WWR, since increased credit deterioration is associated with increased counterparty exposure. The variables Y and ω are normally distributed and independent of each other.

The market factor in (13) is not related to the entity asset value explicitly. The market factor is the key driver of counterparty credit exposure, while the entity asset reflects a counterparty's financial situation. The only assumption is that both the entity asset value and the market factor share the same credit deterioration indicator but with a different correlation structure.

Under a risk-neutral assumption, CVA is expressed as expected credit loss (Hull & White 2012; Rosen & Saunders 2012):

$$\text{CVA} = \int_0^T S_{t|\tau=t}\,\mathrm{d}\,\text{PD}(t \mid Y = y) \tag{14}$$

where LGD $= 1$, $r_t = 0$[3] for simplicity, and the conditional PD is given by (11). Differentiating PD$(t \mid Y = y)$ with respect to y yields:

$$\mathrm{dPD} = \phi\biggl(\frac{C_t - \beta y}{\sqrt{1-\beta^2}}\biggr)\biggl(-\frac{\beta}{\sqrt{1-\beta^2}}\biggr)\,\mathrm{d}y$$

Substituting this result into (14) yields:

$$\text{CVA} = -\frac{\beta F_{0,T}}{\sqrt{1-\beta^2}}\int_{-\infty}^{y_s}\int_{-\infty}^{\infty}\exp\{-\sigma^2 T/2 + \sigma\sqrt{T}(\rho y + \sqrt{1-\rho^2}\,\omega)\} \times \phi(\omega)\phi\biggl(\frac{C_t - \beta y}{\sqrt{1-\beta^2}}\biggr)\phi(y)\,\mathrm{d}\omega\,\mathrm{d}y \tag{15}$$

where $\sigma \equiv \sigma^{\mathrm{F}}_{t,T}$ (given in table A). $C_t = \Phi^{-1}(0.1\%)$ is assumed in this article, which is about three times the asset volatility. This assumption implies the original AAA or AA risk rating entity has been downgraded to a BB grade (table C), from IG to HY. Through some tedious derivations, we obtain:

$$\text{CVA}(\rho) = -\beta F_{0,T}\,\Phi(V_s(\rho))\phi(K_1(\rho)) \tag{16}$$

where:

$$V_s(\rho) = \frac{y_s}{\sqrt{1-\beta^2}} - K(\rho)$$

$$K(\rho) = \frac{\beta C}{\sqrt{1-\beta^2}} + \sigma\sqrt{T}\rho\sqrt{1-\beta^2}$$

$$K_1(\rho) = C - \beta\sigma\sqrt{T}\rho$$

where the correlation coefficient β is negative ($\beta = -0.8063$ in the above example) and the CVA is positive, as shown in table D. The computation is based on two actual oil futures contracts. The first contract is CLJ19 and starts from January 19, 2018. The market price on January 19, 2018 for CLJ19 is $F_{0,T_1} = 58.65$ (delivering in April 2019). The current date is assumed to be March 19, 2018. The maturity time is $T_1 = 1.20$ (years), which is calculated from a contract start date of January 19, 2018 to a delivery date of April 1, 2019. The current time is $t = 0.1890$ (years), which is calculated from the start date to the current date.

The second contract is CLJ20. The parameters are the same as those in the first contract except for $T_2 = 2.20$ and $F_{0,T_2} = 55.99$. Applying this data to (16), the CVA is computed. The results are shown in table D. CVA1 is the base result for $T_1 = 1.20$, $Y_s = 3.15$ (AAA), a one-year transition period (TP $= 1$) and $F_{0,T_1} = 58.65$. CVA2 is the case for a different

[3] *The effect of LGD and interest rate will be discussed separately.*

D. CVA (per contract) vs correlation with varying T, Y_s and TP

ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
CVA1 ($)	0.14	0.22	0.34	0.50	0.72	1.01	1.35	1.76	2.20	2.65	3.09
CVA2 ($)	0.13	0.22	0.37	0.57	0.85	1.21	1.64	2.13	2.63	3.09	3.46
CVA3 ($)	0.15	0.24	0.37	0.56	0.82	1.16	1.58	2.10	2.68	3.31	3.94
CVA4 ($)	0.11	0.17	0.26	0.38	0.52	0.70	0.90	1.12	1.34	1.56	1.71

CVA1, CVA3, CVA4 are for contract CLJ19 with different credit; CVA2 is for contract CLJ20

maturity time $T_2 = 2.20$ and corresponding futures price $F_{0,T_2} = 55.99$, with all other parameters unchanged. CVA3 is the case for a different rating ($Y_s = 3.34$ for an AA rating). CVA4 is the case for a different transition period (TP $= 3$).

The calculation of CVA does not consider the impact of the interest rate in order to focus on market-credit correlation. It is obvious CVA is an increasing function of market-credit correlation ρ, maturity T and the credit deterioration indicator Y_s but a decreasing function of the credit transition period (TP). The computed results can be compared with those from other research. For example, Hull & White (2012) considered a long, single, one-year forward contract of a foreign currency with US$100 million principal. The Hull-White model gives CVA $= 0.548$ with WWR and $\text{CVA}_0 = 0.048$ without WWR. The impact of WWR on CVA is about 11.42 (ie, $\text{CVA}/\text{CVA}_0 = 11.42$). To compare the impact of WWR on CVA with that of the Hull-White model, CVA1 is normalised as $\text{CVA1}_\rho/\text{CVA1}_0 = (1.00, 1.59, 2.45, 3.65, 5.25, 7.30, 9.81, 12.72, 15.92, 19.22, 22.36)$. The value of $\text{CVA1}_\rho/\text{CVA1}_0 = 11.42$ corresponds to $\rho = 0.66$ for the contract CLJ19 with rating AAA. Although the commodities and currencies are different assets and are priced by different models, the impact of WWR on CVA is quite similar.

Conclusions

An analytical expression has been obtained for CVA with WWR for commodity futures. The mark-to-market value of commodity futures is lognormally distributed based on the Gabillon model. The credit deterioration indicator is introduced to consider credit downgrade events. A Gaussian copula model has been employed to specify the correlation between counterparty exposure and credit events. The impact of WWR on CVA is studied explicitly using the obtained formula. The CVA is a function of the credit deterioration indicator, correlation coefficient and asset class. The proposed model captures the impact of WWR on CVA and considers credit granularity by introducing the credit deterioration indicator.

The proposed model may be simple and straightforward, but it illustrates several characteristics that have not been studied before. The model is also compared with a published benchmark model. Further work using Monte Carlo simulation for comparison as well as considering the impact of the volatility assumption can be done. ■

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