Rogue traders versus value-at-risk and expected shortfall

John Armstrong and Damiano Brigo show that, in a Black-Scholes market, value-at-risk and expected shortfall are irrelevant in limiting traders' excessive tail risk-seeking behaviour, as modelled by Kahneman and Tversky's S-shaped utility. The authors argue that to have effective constraints one can introduce a risk limit based on a second, but concave, utility function

In this article, we aim to analyse how classic risk measures such as value-at-risk and expected shortfall (ES) fare in limiting excessive tail risk-seeking behaviour in market players. We will model excessive tail risk-seeking behaviour using Kahneman and Tversky's S-shaped utility (Kahneman & Tversky 1979). For example, S-shaped utility is naturally suited to modelling the behaviour of a trader who cares only about their pay packet and not about the overall loss of the bank, once the latter defaults. A traditional concave utility function would fail to reflect the limited liability of traders.

We point out, via a payoff optimisation approach, that a trader may optimise expected S-shaped utility in the presence of budget constraints regardless of VAR or ES constraints, in the sense that the optimal expected S-shaped utility without VAR or ES constraints will be exactly the same as that with VAR or ES. VAR and ES are thus ineffective in limiting rogue traders. This is particularly important when we consider ES was officially endorsed and suggested as a risk measure by the Basel Committee in 2012 and 2013, partly for its 'coherent risk measure' properties (Acerbi & Tasche 2002; Artzner *et al* 1999).

We then hint at a solution to the problem: it will be enough to introduce a risk limit based on a second utility function, a traditional concave utility, for the expected concave utility constraint to be effective in curbing excessive tail risk-seeking behaviour. Indeed, for this concave utility-based risk constraint the optimum will not be the same as in the case without constraint. We do not prove the result in this article, instead referring the reader to Armstrong & Brigo (2017). The proof given there relies on the ideas used by Hardy and Littlewood to prove their inequality on symmetric decreasing rearrangements (Hardy *et al* 1952).

In this article, we limit our analysis mostly to the Black-Scholes-Merton case (Black & Scholes 1973; Merton 1973), since this is a benchmark model for derivatives valuation and allows us to state our case without excessive mathematical infrastructure and within a familiar setting. However, our result is much more general, and the general theory relies on a result that is similar to the theory of rearrangements behind the Hardy-Littlewood inequality (see Armstrong & Brigo 2017). This is not needed or used here, however, as in this article we use more direct techniques that allow us to avoid such complicated machinery. The result is thus more immediate. The general result follows the same line of research as earlier contributions to behavioural finance, prospect theory and portfolio choice, which we will now review.

Expected utility maximisation under risk measure and budget constraints was considered in Basak & Shapiro (2001), but only under standard utility assumptions, and not an S-shaped utility in particular. In that paper, it is shown that a market player forced by a VAR constraint to reduce portfolio losses in some states would finance these reduced losses by increasing portfolio losses in costly states where the terminal state price density is large. As such states already have the lowest terminal portfolio value for the unconstrained problem, the VAR constraint ends up fattening the left tail of the terminal portfolio distribution. This leads to an increased probability of extreme losses. In Cuoco *et al* (2008), it is shown that VAR constraints play a better role when, as is done in practice, the portfolio VAR is re-evaluated dynamically by incorporating available conditioning information. Again, this is done under standard utility, and S-shaped utility is not considered.

Prospect theory has been studied in relation to risk measures and portfolio choice in a series of papers (He & Zhou 2011; He *et al* 2015; Jin & Zhou 2008; Zhou 2010). These tackle problems similar but not equivalent to the problem we consider here. He & Zhou (2011); Jin & Zhou (2008), for example, do not study risk constraint in optimising S-shaped utilities (with distortions), while in He *et al* (2015) the problem closest to our own is a problem in which expected returns are optimised but expected utility is not. Still, these papers find connections with rearrangements, use law-invariant portfolio optimisation and employ techniques and proofs that deal with and solve a wide range of behavioural finance problems that are similar to those in this article and in our more general paper (Armstrong & Brigo 2017).

The Black-Scholes market

We introduce briefly the Black-Scholes model (Black & Scholes 1973; Merton 1973) for a market with a single risky asset and a bank account. We consider a probability space with a right-continuous filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t : 0 \leqslant t \leqslant T), \mathbb{P})$. In the given economy, two securities are traded continuously from time 0 until time T. The first security, the cash account or bank account, is locally risk-free and its price B_t evolves according to:

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \quad B_0 = 1, \quad \text{with solution } B_t = \mathrm{e}^{rt} \tag{1}$$

where r is a non-negative number. Usually, the risk-free rate r is an $(\mathcal{F}_t)_{t\geqslant 0}$ adapted process but, for simplicity, in this context we assume it is a positive deterministic constant.

As for the second security, given the $(\mathcal{F}_t, \mathbb{P})$-Wiener process W_t, consider the following stochastic differential equation for the price of such a security, typically a stock:

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T \tag{2}$$

with initial condition $S_0 > 0$, and where μ and σ are positive constants. Equation (2) has a unique (strong) solution, given by:

$$S_t = S_0 \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}, \quad 0 \leqslant t \leqslant T \tag{3}$$

We can write the probability law of S_T easily by recalling that W_t is normally distributed with mean zero and variance t. Let us call F_t the cumulative distribution function (CDF) of S_t under the measure $\mathbb{P}$. We have a lognormal distribution for S_t:

$$F_t(y) = \mathbb{P}(S_t \leqslant y) = \mathbb{P}\biggl(W_t \leqslant \frac{\ln(y/S_0) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\biggr)$$
$$= \Phi\biggl(\frac{\ln(y/S_0) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\biggr)$$

where Φ is the CDF of the standard normal distribution.

Further, in the basic Black-Scholes model there are no transaction costs; short selling is allowed without penalty or restrictions, and borrowing and lending occur at the risk-free rate r, with no credit or default risk or funding costs.

The full development of valuation in the presence of counterparty credit risk, liquidity risk, funding and capital costs has been addressed in the literature in recent years: see, for example, Brigo *et al* (2017), a work that tries to stay as close as possible to Black-Scholes while including credit, repurchase agreement and funding effects.

Hence, even if the Black-Scholes model neglects important aspects of valuation, we work under its assumptions because we are interested in S-shaped utility and ES limits in benchmark models. We expect our results to hold in extensions of the basic model, particularly in cases where the resulting valuation approach is very similar to the basic Black-Scholes setting, as in Brigo *et al* (2017).

Simple claims and market price of risk

We consider a simple contingent claim. This is a contract guaranteeing a payoff of the form $\phi(S_T)$ payable at maturity T. If we assume the market is arbitrage-free and complete, the unique no-arbitrage price of our simple claim at time 0 is the expected value:

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-rT}\phi(S_T)]$$

where the expected value is taken under a probability measure $\mathbb{Q}$, the risk-neutral measure, equivalent to $\mathbb{P}$. Under this measure $\mathbb{Q}$:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. It is immediately obvious the ratio S_t/B_t is a $\mathbb{Q}$-martingale, and this is why $\mathbb{Q}$ is sometimes referred to as the martingale measure. We can write the Radon-Nikodym derivative connecting the two equivalent measures as:

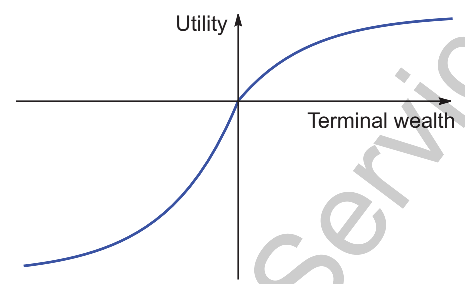
$$Z_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp(-\lambda W_t - \tfrac{1}{2}\lambda^2 t), \qquad \lambda = \frac{\mu - r}{\sigma}$$

In the particular version of the Black-Scholes setting we are working with, this formula can also be written as:

$$Z_t = \frac{\exp(-\frac{1}{2}\lambda^2 t + \lambda(\mu/\sigma - \sigma/2)t)}{S_0^{-\lambda/\sigma}}(S_t)^{-\lambda/\sigma} =: g(S_t)$$

The Radon-Nikodym derivative Z_t becomes an explicit function of the underlying S_t. This makes both the Radon-Nikodym derivative and the payoff functions of the same variable S_T and renders the analysis of their interaction explicit.

1 Typical example of an S-shaped utility curve



The constant λ is called the market price of risk or a particular version of the Sharpe ratio. We have:

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-rT}\phi(S_T)] = \mathbb{E}^{\mathbb{P}}\biggl[\mathrm{e}^{-rT}\phi(S_T)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_T}\biggr]$$
$$= \mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-rT}\phi(S_T)g(S_T)] \qquad (4)$$

We assume $\lambda > 0$, ie, $\mu > r$. This means a trader will be interested in investing in stock S, since its expected return will exceed the risk-free rate in the market.

Standardisation to a uniform risky asset

In our subsequent utility analysis, we will rescale the risky asset's probability distribution to be uniform. In the Black-Scholes context, this means that instead of expressing simple claims as functions of S_T, we will express them as functions of $X := F_T(S_T)$. Indeed, we know that X has a standard uniform distribution. This allows us to write the price of the claim as:

$$V_0 = \mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-rT}\phi(S_T)g(S_T)] = \mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-rT}\phi(F_T^{-1}(X))g(F_T^{-1}(X))]$$
$$= \mathrm{e}^{-rT}\int_0^1 f(x)q(x)\,\mathrm{d}x$$

where:

$$f(x) = \phi(F_T^{-1}(x)), \qquad q(x) = g(F_T^{-1}(x))$$

Given our expressions above for F_T, g and our assumption on λ, it is clear q is decreasing. Further, a simple limit calculation based on the fact $g(S)$ is essentially a power to the $-\lambda$ shows:

$$\lim_{x \to 0^+} q(x) = +\infty \quad \text{if } \lambda > 0$$

S-shaped utility and tail risk-seeking behaviour

It is observed in Kahneman & Tversky (1979) that individuals appear to have preferences governed by an S-shaped utility function, u. This is increasing, strictly convex on the left, strictly concave on the right, non-differentiable at the origin and asymmetrical: negative events are considered worse than positive events, which are considered good.

A typical S-shaped utility function is shown in figure 1.

Whether the cause of S-shaped utility functions is the irrationality or limited liability of a market player, there is certainly good evidence they are a useful tool for modelling real-world behaviour. A regulator or risk manager could certainly consider them as a possibility.

As not all of the characteristics of S-shaped utility functions are important to us, we adopt a slightly more general definition. Suppose we have a candidate utility function as an increasing function $u(w)$, which represents the utility of holding the wealth w. Inspired by the definition of S-shaped utility in Kahneman & Tversky (1979), we assume the following estimates hold:

$$\begin{aligned} u(w) &\geqslant -c|w|^{\eta} && \text{for } w < N_{\mathrm{R}} < 0,\ \eta \in (0,1) \\ u(w) &\leqslant k|w|^{\beta} && \text{for } w > N_{\mathrm{I}} > 0,\ \beta > 1 \end{aligned}$$

for some positive constants c, k. We call such a utility function risk seeking in the left tail and risk averse in the right tail.

In this work, we finally define formally an S-shaped utility curve $u(w)$ as an increasing function that is negative for $w < 0$, positive for $w > 0$, concave for $w > 0$, risk seeking in the left tail and risk averse in the right tail.

Utility maximisation under budget and expected shortfall constraints

We are now interested in a utility maximisation problem in the Black-Scholes market. The trader or investor wishes to optimise over all simple claims ϕ to find the claim that gives them the maximum utility:

$$\sup_{\phi} \mathbb{E}^{\mathbb{P}}[u(\phi(S_T))] \tag{5}$$

under constraints:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-rT}\phi(S_T)g(S_T)] &\leqslant C && \text{(budget)} && (6) \\ \mathrm{ES}(p,T,\phi) &\geqslant L_0 && \text{(ES constraint)} && (7) \end{aligned}$$

where $\mathrm{ES}(p,T,\phi)$ denotes the expected shortfall of $\phi(S_T)$ over the horizon T at confidence level p. In this formulation of the problem, we implicitly assume the additional constraint that the expectation and ES are both well defined and finite. We can reformulate the above problem after uniform rescaling $X = F_T(S_T)$:

$$\sup_f \int_0^1 u(f(x))\,\mathrm{d}x \tag{8}$$

under constraints:

$$\begin{aligned} \mathrm{e}^{-rT}\int_0^1 f(x)q(x)\,\mathrm{d}x &\leqslant C && \text{(budget)} && (9) \\ \frac{1}{p}\int_0^p f(x)\,\mathrm{d}x &\geqslant L_0 && \text{(ES constraint)} && (10) \end{aligned}$$

The ES representation in this last formulation comes from Acerbi & Tasche (2002, (3.3)). Again, we implicitly assume the latter two integrals exist and are finite.

We will now show that, under the assumption $\lambda > 0$ (recall that q depends on λ), the ES constraint is not relevant in that the maximum attained under the constraint is the unconstrained $\sup_x u(f(x))$. Thus, for S-shaped utility functions that are risk seeking in the tail, as we expect from traders, the ES constraint is ineffective in curbing excessive risk taking.

First, we could consider all constant functions f and optimise on those. If we denote by k the function constantly equal to k, the optimisation problem is:

$$\begin{aligned} &\sup_k u(k) && \text{(under constraints)} && (11) \\ &k \leqslant \mathrm{e}^{rT}C && \text{(budget)} && (12) \\ &k \geqslant L_0 && \text{(ES constraint)} && (13) \end{aligned}$$

So, we know that if u^* is the optimal utility in the full problem, we have as our lower bound the result for the 'constant f' problem:

$$u^* \geqslant \sup_{y \in [L_0, \mathrm{e}^{rT}C]} u(y)$$

As it turns out, it is enough to move to the next simplest possible function f, namely a two-step piecewise constant function, to obtain a much sharper result.

THEOREM 1 (Irrelevance of ES constraint in S-shaped utility maximisation in a Black-Scholes market) *Consider a Black-Scholes market in which the bank account price B and the risky asset (stock) price S follow the differential equations:*

$$\begin{aligned} \mathrm{d}B_t &= rB_t\,\mathrm{d}t, \quad B_0 = 1 \\ \mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 \end{aligned} \tag{14}$$

where W is a standard Brownian motion under the measure $\mathbb{P}$ in a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and σ is a positive deterministic constant modelling the volatility. Assume $\lambda = (\mu - r)/\sigma > 0$, and assume we are given an S-shaped utility function u that is risk seeking in the tail; namely, there are constants N and η such that:

$$u(w) \geqslant -c|w|^{\eta} \quad \text{for } w < N,\ \eta \in (0,1) \tag{15}$$

Let U be the supremum over claims f defined by:

$$\sup_f \int_0^1 u(f(x))\,\mathrm{d}x$$

under constraints:

$$\begin{aligned} \mathrm{e}^{-rT}\int_0^1 f(x)q(x)\,\mathrm{d}x &\leqslant C \\ \frac{1}{p}\int_0^p f(x)\,\mathrm{d}x &\geqslant L_0 \end{aligned}$$

Then $U = \sup_y u(y)$.

PROOF We will use non-decreasing piecewise constant functions taking only two values. Given three constants k_1, k_2, α, with $0 < \alpha < p$ and $k_1 > k_2$, define:

$$f(x; k_1, k_2, \alpha) := k_2 1_{\{x<\alpha\}} + k_1 1_{\{x \geqslant \alpha\}}$$

We will omit the arguments k and α for brevity in this proof. Consider the expected utility for this function:

$$\mathbb{E}^{\mathbb{P}}[u(f(X))] = \int_0^1 u(f(x))\,\mathrm{d}x = \alpha u(k_2) + (1-\alpha)u(k_1) \tag{16}$$

Let us now write the two constraints in the optimisation problem: the budget constraint and the ES constraint. The budget constraint reads:

$$\int_0^{\alpha} f(x)q(x)\,\mathrm{d}x + \int_{\alpha}^1 f(x)q(x)\,\mathrm{d}x \leqslant \mathrm{e}^{rT}C$$

or

$$k_2 \int_0^{\alpha} q(x)\,\mathrm{d}x + k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x \leqslant \mathrm{e}^{rT} C$$

The ES constraint reads:

$$\frac{1}{p}\biggl[\int_0^{\alpha} f(x)\,\mathrm{d}x + \int_{\alpha}^{p} f(x)\,\mathrm{d}x\biggr] \geqslant L_0$$

or

$$\frac{1}{p}[\alpha k_2 + (p-\alpha)k_1] \geqslant L_0$$

Putting both constraints together, we obtain:

$$\frac{pL_0 - (p-\alpha)k_1}{\alpha} \leqslant k_2 \leqslant \frac{C\mathrm{e}^{rT} - k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x}{\int_0^{\alpha} q(x)\,\mathrm{d}x}$$

In this constrained interval, we now pick a special point, the initial point, thus assuming the ES constraint is met as an equality:

$$k_2 := \frac{pL_0 - (p-\alpha)k_1}{\alpha}$$

We need to check this constraint is consistent with our assumption that $k_1 > k_2$. This holds as long as k_1 is chosen to be sufficiently large and positive, namely $k_1 > (L_0)^+$. We denote the related function f as:

$$\bar{f}(x; k_1, \alpha) := f(x; k_1, (pL_0 - (p-\alpha)k_1)/\alpha, \alpha)$$

and the related expected utility (16) specialises to:

$$\mathbb{E}^{\mathbb{P}}[u(\bar{f}(X))] = \alpha u\biggl(\frac{pL_0 - (p-\alpha)k_1}{\alpha}\biggr) + (1-\alpha)u(k_1) \tag{17}$$

The budget constraint for $\bar{f}$ becomes:

$$pL_0 - (p-\alpha)k_1 \leqslant \frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}\biggl(C\mathrm{e}^{rT} - k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x\biggr) \tag{18}$$

We can obtain a lower limit to the right-hand side of the budget constraint (18) by noting:

$$\frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}\biggl(C\mathrm{e}^{rT} - k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x\biggr) > \frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}(C\mathrm{e}^{rT} - k_1)$$

If we now impose $C\mathrm{e}^{rT} < k_1 < M_1$ for a sufficiently large positive constant M_1, then the right-hand side becomes negative and we can estimate it further with:

$$\frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}(C\mathrm{e}^{rT} - k_1) > \frac{1}{q(\alpha)}(C\mathrm{e}^{rT} - k_1) > \frac{1}{q(\alpha)}(C\mathrm{e}^{rT} - M_1)$$

where the first estimate follows from q being decreasing, so if we take $\alpha \leqslant \varepsilon_\alpha$ for a positive $\varepsilon_\alpha > 0$, we conclude:

$$\frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}\biggl(C\mathrm{e}^{rT} - k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x\biggr) > \frac{1}{q(\varepsilon_\alpha)}(C\mathrm{e}^{rT} - M_1)$$

The range for ε_α will depend on M_1. However, we can estimate the left-hand side of the budget constraint (18) from above by noting:

$$pL_0 - (p-\alpha)k_1 \leqslant pL_0 - (p-\varepsilon_\alpha)k_1$$

and this can be made sufficiently negative, say less than -1, by choosing a sufficiently large k_1:

$$k_1 \geqslant \frac{pL_0 + 1}{p - \varepsilon_\alpha}$$

where we are further assuming $\varepsilon_\alpha < p$. Thus, we may write:

$$\begin{aligned} pL_0 - (p-\alpha)k_1 &\leqslant pL_0 - (p-\varepsilon_\alpha)k_1 \leqslant -1 \\ -1 &\leqslant \frac{1}{q(\varepsilon_\alpha)}(C\mathrm{e}^{rT} - M_1) \\ &\leqslant \frac{\alpha}{\int_0^{\alpha} q(x)\,\mathrm{d}x}\biggl(C\mathrm{e}^{rT} - k_1 \int_{\alpha}^{1} q(x)\,\mathrm{d}x\biggr) \end{aligned} \tag{19}$$

as long as we make sure the central inequality:

$$-1 \leqslant \frac{1}{q(\varepsilon_\alpha)}(C\mathrm{e}^{rT} - M_1) \tag{20}$$

holds. Since the denominator tends to $+\infty$ as ε tends to 0, it suffices, again, to choose a small enough ε_α.

We conclude that both the budget and ES constraints are met if we require k_1 and α satisfy:

$$m_1 < k_1 < M_1, \quad \alpha < \varepsilon_\alpha \tag{21}$$

with the lower bound satisfying:

$$m_1 \geqslant \max\biggl(C\mathrm{e}^{r}T, L_0, \frac{pL_0 + 1}{p - \varepsilon_\alpha}\biggr)$$

with constants $M_1 > 0$ large enough and $\varepsilon_\alpha > 0$ small enough, and with ranges determined by the other constants C, r, T, L_0, λ. We assume these hold from now on.

We now go back to our expected utility (17). Given the estimate we obtained in (19), we can write:

$$\frac{pL_0 - (p-\alpha)k_1}{\alpha} \leqslant \frac{-1}{\alpha} < \frac{-1}{\varepsilon_\alpha}$$

We may now invoke inequality (15) for the utility function to deduce:

$$u\biggl(\frac{pL_0 - (p-\alpha)k_1}{\alpha}\biggr) \geqslant -c\,\biggl|\frac{pL_0 - (p-\alpha)k_1}{\alpha}\biggr|^{\eta}$$

as long as $\alpha \leqslant \varepsilon_\alpha$ with ε_α chosen to be sufficiently small. We can use this to estimate the expected utility in (17) as follows:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[u(\bar{f}(X))] &= \alpha u\biggl(\frac{pL_0 - (p-\alpha)k_1}{\alpha}\biggr) + (1-\alpha)u(k_1) \\ &\geqslant -\alpha c\,\biggl|\frac{pL_0 - (p-\alpha)k_1}{\alpha}\biggr|^{\eta} + (1-\alpha)u(k_1) \end{aligned}$$

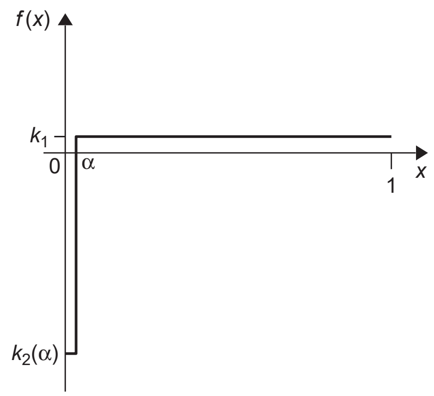
Given that $0 < \eta < 1$, we can conclude that for $\alpha \to 0^+$ the first term on the right-hand side tends to zero, and we are left with:

$$\lim_{\alpha \to 0^+} \mathbb{E}^{\mathbb{P}}[u(\bar{f}(X; k_1, \alpha))] = u(k_1)$$

with the budget and ES constraints satisfied implicitly via (21) when taking the limit.

It follows that our general utility optimisation problem has the solution $u(k_1)$ when optimising in the subclass of admissible functions $\bar{f}$. It also follows that the optimum on a larger class will yield an optimal expected utility larger than $u(k_1)$ for all possible k_1. However, it is obvious the optimal expected utility will be bounded from above by the supremum of the utility function. This then proves the claim that the optimal expected utility will be equal to the supremum of the utility function. □

2 Payoff used in the proof of the main theorem



It may be interesting to check what the limiting function $x \mapsto \bar{f}(x; k_1, \alpha)$ looks like for small α. Recall that:

$$\begin{aligned}\bar{f}(x; k_1, \alpha) &:= \frac{pL_0 - (p-\alpha)k_1}{\alpha} 1_{\{x<\alpha\}} + k_1 1_{\{x \geqslant \alpha\}} \\ &=: k_2(\alpha) 1_{\{x<\alpha\}} + k_1 1_{\{x \geqslant \alpha\}}\end{aligned}$$

Fixing k_1, for very small α the first constant becomes negative and very large in absolute value, but on a very small interval $x \in [0, \alpha)$; the second constant, however, is equal to k_1 on a large interval $x \in [\alpha, 1]$. We thus have a digital option with an extremely negative constant payoff in a small range of the rescaled uniform underlying, $[0, \alpha)$, and with a much smaller positive payoff in the remaining range $[\alpha, 1]$. This is illustrated in figure 2.

An effective risk constraint based on a second concave utility

The focus of this short article is the negative result above. However, we would like to hint at a possible solution for the ineffectiveness of the VAR and ES constraints; this is developed fully in Armstrong & Brigo (2017).

Above, we have seen a result related to the following fact: an investor with S-shaped utility function u_{I}, with $\lim_{x \to +\infty} u_{\mathrm{I}}(x) = +\infty$ plus a budget constraint, and who is subject only to ES (or VAR) constraints for risk can find a sequence of portfolios satisfying these constraints that has an expected u_{I}-utility tending to infinity. This implies VAR or ES constraints cannot limit excessive tail risk-seeking behaviour.

If the risk constraint is based instead on a second utility function of the type:[1]

$$u_{\mathrm{R}}(x) = -(-x)^{\gamma_{\mathrm{R}}} 1_{\{x \leqslant 0\}}$$

with $\gamma_{\mathrm{R}} > 1$, where the constraint requires the payoff ϕ to be in the set:

$$\{\phi \colon \mathbb{E}[u_{\mathrm{R}}(\phi(S_T))] \geqslant L\}$$

for a negative loss level L, then any sequence of portfolios whose expected S-shaped utility u_{I} tends to $+\infty$ will have expected u_{R} tending to $-\infty$, and thus will not be acceptable for the risk constraints. In this sense, a classic concave utility u_{R} adopted by the risk manager or regulator can be used more effectively than VAR or ES in limiting excessive tail risk-seeking behaviour in the presence of limited liability/S-shaped utility investors. ■

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[1] *We use the term 'utility function' for u_{R} somewhat informally here, as what is important for our result is the functional form of the limit set and not whether it has been derived from any specific individual's utility function. One might loosely think of u_{R} as the regulator's or risk manager's utility function, although in practice the regulator or risk manager should choose any risk limits to reflect the risk preferences of whoever bears the risk. We have not attempted to consider how they should do this. We are simply assuming one of the limits set has the given functional form.*

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