Local volatility from American options

Stefano De Marco and Pierre Henry-Labordère focus on short-time asymptotics for American options in the case of local and stochastic volatility models. As a by-product, they obtain an efficient algorithm for calibrating Dupire’s local volatility to American options, starting from an arbitrage-free parameterisation of an European-implied volatility.

**Reminders on American options.** Let \( u(t, x, a) \) denote the price at time \( t \) of the American put with maturity \( T \) and strike \( K \), while \( x^T(t, a) \) denotes the option exercise boundary, which depends on time and volatility. \( (u, x^T) \) is the well-known unique solution to the parabolic obstacle PDE:

\[
\begin{align*}
\partial_t u + \mathcal{L} u - ru &= 0, \quad t < T, x > x^T(t, a) \\
 u(t, x, a) &= (K - x)^+, \quad t < T, x \leq x^T(t, a) \\
 u(t, x, a) &= (K - x)^+, \quad t = T, x \in (0, \infty)
\end{align*}
\]

where \( \mathcal{L} \) is the infinitesimal generator associated with the Markov diffusion \( (X_t, \alpha_t) \). In particular, \( u \) satisfies the continuation condition:

\[
u(t, x^T(t, a), a) = K - x^T(t, a) \tag{3}
\]

and the smooth-fit condition:

\[
\alpha(x^T(t, a), a) = -1
\]

for all \((t, a)\) with \( t < T \). The Feynman-Kac representation of the solution to (2) provides the early premium formula for the American put option (see, for example, Carr et al 1992):

\[
u(t, x, a) = \mathbb{E}_t, x, a \left[ e^{-r(T-t)}(K - X_T)^+ \right] + rK \int_t^T \mathbb{E}_s, x, a \left[ e^{-r(s-t)}1_{X_s \leq x^T(s, a)} \right] ds \tag{4}
\]

This formula states the price of the American put is the same as the price of the European put plus an extra term, which quantifies the premium associated with having the right to exercise the option earlier than maturity. Note since \((X_t, \alpha_t)\) is a homogeneous Markov process, we have:

\[
u(t, x, a) = \bar{u}(T-t, x, a), \quad x^T(t, a) = \tilde{x}(T-t, a)
\]

**Notation.** We use \( p(t, y, b \mid x, a) \) to denote the transition density of the process \((X, a)\) from the point \((x, a)\) to the point \((y, b)\) over time \( t \). Moreover, \( P_{t, x, a}(T, K) = \mathbb{E}_{t, x, a}[e^{-r(T-t)}(K - X_T)^+] \) (respectively, \( C_{t, x, a}(T, K) \)) denotes the price of the European put (call) with maturity \( T \) and strike \( K \), and \( F_{t, x, a}(T, y) = \mathbb{E}_{t, x, a}[X_T \mid \mathcal{F}_{t, x, a}] \) is the cumulative distribution function of \( X_T \mid \mathcal{F}_{t, x, a} \). For simplicity, whenever the initial condition is fixed and clear from the context, we may omit the dependence with respect to \((t, x, a)\) and write \( P(T, K), C(T, K) \) and \( F(T, y) \).

From (3), we have that \( \tilde{x}(\cdot, \cdot) \) is a solution of:

\[
K - \tilde{x}(T-t, a) = \int_{\mathbb{R}_+^2} e^{-r(T-t)}(K - y)^+ p(T-t, y, b \mid \tilde{x}(T-t, a), a) dy db + rK \int_t^T ds \int_{\mathbb{R}_+} db \tilde{x}(s, t, a) \times p(s-t, y, b \mid \tilde{x}(T-t, a), a)
\]

**A representation formula for the exercise boundary.** Consider a homogeneous stochastic volatility model (SVM):

\[
dX_t = rX_t dt + a_t \sigma(X_t) dW_t
\]

where the stochastic volatility \((a_t)_t \geq 0\) is the solution of the SDE:

\[
da_t = \beta(a_t) dt + \nu(a_t) dW'_t, \quad a_0 = \alpha > 0, \quad d[W, W']_t = \rho dt
\]

We assume standard Lipschitz conditions for \( \beta \) and \( \nu \), with \( \nu(0) = 0 \), so \( a_t \geq 0 \) for all \( t \). Further, the local volatility function \( \sigma(t) \) satisfies \( \sigma \in C^2(\mathbb{R}_+) \), and there exists a positive constant \( m > 0 \) such that:

\[
\frac{1}{m} x \leq \sigma(x) \leq mx, \quad |\sigma'| \leq m, \quad |x \sigma''(x)| \leq m
\]

for all \( x > 0 \). This ensures (1) has a unique strictly positive solution \( X_t \).
Finding the exercise boundary (even in the short-maturity regime $\tau := T - t \to 0$) is not an easy task. We propose a new formula that greatly simplifies this problem.

### A new early premium formula.

**Proposition 1** Let $x(\cdot, \cdot)$ be a function in $C^{1,0}(0, T) \times \mathbb{R}^+$ and take $u$ as given in (4). Then, $(u, x^2)$ is the unique solution of (2) if and only if, for all $\tau > 0$, $x(\tau, a) = x^2(T - \tau, a)$ satisfies:

$$
e^{-\tau T} p(T, a) \int_{\mathbb{R}^+} p(\tau, x) d\tau = 2R \int_{\mathbb{R}^+} e^{-\tau x} p(\tau, x - u, b \mid \tau, a) \cdot x^2(\tau, a) d\tau$$

where $x^2(\cdot, \cdot)$ denotes the derivative of $x$ with respect to the first variable.

For the proof of proposition 1, we direct the reader to De Marco & Henry-Labordère (2016).

**With repurchase agreement.** The representation of the exercise boundary in proposition 1 can be extended to include a constant repurchase agreement (repo)/dividend rate $q$:

$$dX_t = \frac{r - q}{\sigma(X_t)} dW_t$$

**Asymptotic behaviour of the exercise boundary.** We derive an asymptotic expansion of the exercise boundary using the integral representation (5) obtained in the previous section. We let $x(t) = x^2(T - \tau)$ denote the exercise boundary as a function of time to maturity $\tau$.

**Theorem 1** As a function of time to maturity $\tau$, the put exercise boundary $x(t)$ satisfies:

$$2 \gamma^\tau = \tau \ln \left[ \left( 1 + \frac{1}{\ln(\gamma)} \right)^2 + O \left( \frac{1}{(\ln(\gamma))^3} \right) \right]$$

as $\tau \to 0$, where $\gamma = \sigma(K)^2 / 8\pi\tau(K)^2$.

Note the constant $\gamma$ is homogeneous to (volatility/interest rate)$^2$ and hence to time. Note also the error term in (8) is $O(1/\ln(\tau)^3)$.

In the proof of theorem 1 reported in De Marco & Henry-Labordère (2016), we have made use of the following classical result on the short-time behaviour of the transition density of $X$:

$$p(t, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x - \psi)^2}{2\tau}}(1 + O(1))$$

where a second-order term in $1/\ln(\gamma)$ appears.

### Local volatility models

We now consider a homogeneous local volatility model for which $a_t = 1$:

$$dX_t = r X_t + \sigma(X_t) dW_t$$

**Asymptotic behaviour of the exercise boundary.** We derive an asymptotic expansion of the exercise boundary using the integral representation (5) obtained in the previous section. We let $x(t) = x^2(T - \tau)$ denote the exercise boundary as a function of time to maturity $\tau$.

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where a second-order term in $1/\ln(\gamma)$ appears.

$^1$ This hinges on the way we write the right-hand side of (8). Applying Taylor’s theorem around $t = 0$, the factor $\ln[1/\ln(\gamma)]$ in (8) admits the expansion:

$$\ln(\gamma) = \frac{2}{\ln(\gamma)} + O \left( \frac{1}{(\ln(\gamma))^2} \right)$$

where a second-order term in $1/\ln(\gamma)$ appears.
When \( a_i = 1 \), the mirror models are defined by:

\[
\begin{align*}
\text{I:} & \quad dX_t = (r - q) X_t dt + \sigma (X_t) dW_t \\
\text{II:} & \quad dX_t = (q - r) X_t dt - \sigma (X_t) dW_t 
\end{align*}
\]

Note in model II the process \( X_t \) is expressed in units of 1/cash. The following relation between the call exercise boundary in model I and the put exercise boundary in model II is easily derived: \( T^{-2} X(t) = (X_{t1}^2; K(t))^{-1} \).

Using theorem 3.4 in De Marco & Henry-Labordère (2016) to handle non-zero repo rates, this implies the following corollary:

**Corollary 2 (Exercise boundary near expiry for call options)** Let \( x(t) > K \) be the exercise boundary of the American call in the local volatility model I above as a function of time to maturity \( \tau = T - t \).

**Case** \( q < r \):

\[
\hat{x}(\tau) = \frac{rK}{q} \left( 1 - q \frac{rK}{q} \right) \sqrt{2\tau } \\
+ \frac{qK}{rK} \sigma^2 \left( \frac{qK}{rK} \right) \left( \frac{qK}{rK} \sigma(qK) \right) \tau^{-1} + O(\tau^{3/2})
\]

**Discrete dividends.** In this section, we assume the spot process \( X_t \) falls by the dividend amount \( D_i(X_{t_i}^\tau) = \beta_i X_{t_i}^\tau \), with \( \beta_i \in (0,1) \), paid at the dates \( 0 < t_0 < \cdots < t_1 < T \), and that between dividend dates it follows a local volatility model. The dynamics of \( X \) then read:

\[
X_t = \int_0^t r X_s ds + \int_0^t \sigma (X_s) dW_s - \sum_{t_i < t} D_i (X_{t_i}^\tau)
\]

Note that, due to the jump in the trajectories of the underlying process at \( t_i \), the exercise boundary will fail to be monotonic or continuous on \([0, T] \) (cf, Göttsche & Vellekoop (2011) and figures 3 and 4). On the interval \([t_1, T]\) after the last dividend date, we can still use the approximation of \( x(t) \) given in corollary 1.
Following Jourdain & Vellekoop (2011), in the Black-Scholes model the exercise boundary \( \sigma(x) \equiv \sigma(x) \) is right-continuous, decreasing on some interval \([t_i^*, t_i]\) with \(t_i+1 < t_i^* < t_i\), and converging to zero as \(t \to t_i^*\); it is also (slightly increasing, but) essentially constant on \([t_i+1, t_i^*]\) (see figures 3 and 4).

We will use this information as an ansatz on the shape of the exercise boundary in (11).\(^2\) We compute an explicit approximation by considering the early premium formula (4), that is, we set \(\sigma(t) \equiv 0\). This is a brutal approximation, but, as we will show below, it already provides accurate values for the exercise boundary and the option price (see figures 3 and 4).

When \(\sigma\) is set to zero, conditional on \(X_t = x\), we have \(X_t = x e^{(r-t)}(B(s)/B(t))\) for all \(s \geq t\), where:

\[
B(t) = \prod_{i: t_i \leq T} (1 - \beta_i)
\]

(and \(B(t) = 1\) for \(t < t_0\)). Using the continuation condition \(u(t, x(t)) = K - x(t)\), the zero-noise approximation of (4) reads:

\[
K - e^{rT} B(t) y(t) = e^{-r(T-t)} (K - y(t)) e^{rT} B(T) + rK \int_0^T e^{-r(T-s)} y(s) \, ds
\]

(12)

where \(y(s) := x(s)/(B(s)e^{rT})\) for \(s \geq t\). From this equation, we obtain the following approximation for \(x(t)\) between the dividend dates \(t_i+1\) and \(t_i\) (in particular, \(x(t_i^*) = 0\), as expected):

\[
x(t) = \begin{cases} 
1 - e^{r(t_i - t)} / \left(1 - (B(T)/B(t))\right) & t \in [t_i^*, t_i), \\
e^{r(t_i^* - t)} / \left(1 - (B(T)/B(t))\right) & t \in [t_{i+1}, t_i^*)
\end{cases}
\]

(13)

(14)

where:

\[t_i^* = \max \left(t_{i+1}, t_i + \frac{1}{r} \ln \left(1 - \left(1 - \frac{B(T)/B(t_{i+1})}{K}\right) \right)\right)\]

Adding a repo rate \(q < r\) consists of replacing \(r\) with \(r - q\). When \(n = 1\), we can compare (13) with the exact asymptotics \(x(t) \sim (rK/\beta_1)(t_1 - t)\) as \(t \to t_1^*\) from Jourdain & Vellekoop (2011, theorem 3.6).

\section*{Reproduction of American from (synthetic) European vanillas.}

Given an (arbitrage-free) European implied volatility surface \(\sigma^Eur(\cdot, \cdot)\), we set \(P(T, K) := P_{BS}(T, K, \sigma^Eur(\cdot, \cdot))\), where \(P_{BS}(T, K, \sigma)\) denotes the Black-Scholes formula for a European put option with strike \(K\) and maturity \(T\).

We are going to 'reparameterise' the results of the previous section in terms of \(\sigma^Eur\). Notably, we will look for a formula relating the American exercise boundary to the European implied volatility. Let us start from the case with no dividends treated previously.

\section*{Solution I.}

If the implied volatility \(\sigma^Eur(T, K)\) is perfectly calibrated by a local volatility model \(\sigma(T, K)\), then it follows from Berestycki et al (2002) that:

\[
\lim_{T \to 0} \sigma^Eur(T, K) = \frac{\ln(K/X_0)}{\int_{X_0}^{K} dz/\sigma(T, z)} |_{T = 0}.
\]

We will use (15) as an approximation of \(\sigma^Eur(T, K)\) when \(T\) is small. Inverting the equation with respect to the local volatility \(\sigma\), we obtain:

\[
\sigma(T, K) = \frac{K \sigma^Eur(T, K)}{1 - K \ln(K/X_0) \sigma_X \ln \sigma^Eur(T, K)}
\]

We can now rewrite the exercise boundary in terms of the European smile \(\sigma^Eur\) (instead of the model \(\sigma(t)\)). Writing:

\[
\int_0^K Kdz/\sigma(T, z) = \int_0^K dz/\sigma(T, z) - \int_0^K dz/\sigma(T, z)
\]

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inside (8), and then injecting (15) and (16), we obtain the following approximation of the put exercise boundary close to maturity:

$$\frac{\ln(K/X_0)}{\sigma^{\text{Eur}}(T, K)} - \frac{\ln(x_{IV}^{T,K}(t)/X_0)}{\sigma^{\text{Eur}}(T, x_{IV}^{T,K}(t))} = \sqrt{\frac{r}{\tau} \left(1 - \frac{2}{\ln(r/\gamma)}\right)}$$  (17)

with

$$\gamma = \frac{\mu^2}{8 \pi \tau^2} \quad \mu = \frac{\sigma^{\text{Eur}}(T, K)}{1 - K \ln(K/X_0) \partial_K \ln \sigma^{\text{Eur}}(T, K)}$$

Equation (17) is implicit in $x_{IV}^{T,K}$, replacing $\sigma^{\text{Eur}}(T, x_{IV}^{T,K}(t))$ with $\sigma^{\text{Bust}}(T, K)$ on the left-hand side, we obtain an explicit approximation for the exercise boundary.

**Solution 2.** We evaluate the local volatility from Dupire’s formula. Let:

$$\sigma^{\text{Eur}}(T, x) = \sigma^{\text{Eur}}(T, X_0 e^{rT + x})$$

denote (with a slight abuse of notation) the implied volatility as a function of forward log strike. We have:

$$\sigma(T, K)^2 = \frac{(\sigma^{\text{Eur}})^2 + 2T \sigma^{\text{Eur}} \partial_T \sigma^{\text{Eur}}}{T \sigma^{\text{Eur}} (\sigma^{\text{Eur}})'^2 - (T^2 (\sigma^{\text{Eur}})^2) (\sigma^{\text{Eur}})'^2} + (1 - (\sigma^{\text{Eur}})'/\sigma^{\text{Eur}})^2 = \log(K e^{-rT}/X_0)$$  (18)

The extension of theorem 1 and corollary 1 to the time-inhomogeneous case yields:

$$x_{IV}^{T,K}(t) = \frac{K}{\sigma(T, K)} \sqrt{\frac{r}{\tau} \left(1 - \frac{2}{\ln(r/\gamma)}\right)}$$  (19)

with $\gamma = \sigma(T, K)/8 \pi \tau^2 K^2$, where we use the (explicit) local volatility given by (18).

Let $A(T, K)$ be the price of an American put option at $t = 0$. Using (17) or (19) and the early premium formula (4), we can write, for all $T > 0$:

$$A(T, K) = P(T, K) + r K \int_0^T \sigma^{\text{Eur}}(s, x_{IV}^{T,K}(s)) ds$$  (20)

This expression provides a static hedging of American put options with vanillas. It also gives an explicit map between a (synthetic) European implied volatility $\sigma^{\text{Eur}}$ and American options (within a Dupire local volatility model, as $T$ is small). We exploit this map below to design a fast calibration of the local volatility on American options. In the presence of continuous dividends/repo or call options, the equations corresponding to (17)–(19) and (20) can be derived from the early premium formula with repo rates (De Marco & Henry-Labordère 2016, equation (2.19)), as well as De Marco & Henry-Labordère (2016, theorem 3.4) and corollary 2.

For discrete dividends, we plug the approximations (17) or (19) into the early premium formula (20) between the last dividend date and the option maturity, and the approximations ((13), (14)) between dividend dates.

Later, we will illustrate how the approximation (20) is applicable even when the maturity is around two years (American options are typically not liquid beyond this).

**Calibration of the local volatility on American options: recipe.**

(i) We start with an arbitrage-free parameterisation of a (synthetic) European implied volatility $\sigma^{\text{Bust}}(T, K)$ that depends on parameters denoted by $\hat{\lambda}$. For example, one can use surface stochastic volatility inspired parameterisations (Gatheral & Jacquier 2014). By using ((17), (20)) or ((19), (20)) and optimising these parameters, we can calibrate American options:

$$\text{Find } \hat{\lambda} : \sigma^{\text{Bust}}(T, K) \rightarrow A(T, K) = A^{\text{mk}}(T, K)$$

(ii) Compute the local volatility $\sigma(\cdot, \cdot)$ from $\sigma^{\text{Bust}}(\cdot, \cdot)$ using Dupire’s formula (18).
Note that no PDE solver or parametric local volatility are needed. The only approximation is the computation of $\frac{1}{\sqrt{\mathbb{E} T}}$ using (17) or (19).

### Homogeneous SVM

Theorem 1 can be extended to the setting of a homogeneous SVM such as (1) using short-time asymptotics for the density of SVMs (see Henry-Labordère 2008, chapter 6). The results in this framework are reported in De Marco & Henry-Labordère (2016, section 4), where an asymptotic formula for the put exercise boundary in the stochastic alpha, beta, rho (SABR) model is provided.

### Numerical experiments

Formulas (17), (19) and (20) are used in our numerical experiments. Results for (17) and (19) did not show any sizeable differences.

- **Black-Scholes model.** We have checked the validity of (20) in the case of a Black-Scholes model, i.e., $\sigma(t)$ $\equiv$ $\sigma$ $x$, and we have used $\sigma$ $=$ 0.5, $r$ $=$ 0.04 and $T$ $=$ 1.5 years.

In figure 1, we compare the prices of put options with strikes ranging from 0.2 to 1 (spot = 1) obtained using a PDE pricer and formula (20). Prices are quoted in terms of the European implied volatility, i.e., the number that, when put into the Black-Scholes formula for a put option with strike $K$ and maturity $T$, reproduces the prices of an American put option (we made this choice only for display purposes). We used the approximation of the exercise boundary (17)–(19) at both the zero and first order in $O(1/\ln r)$ (figure 1(b)). The result for the option price is almost exact when using the first-order approximation. The validity of our approximation for the exercise boundary deteriorates when $r$ is small for out-of-the-money (OTM) puts. As a conclusion, (20) is sufficiently precise to be used to infer an American implied volatility, i.e., the number that, when put in (20), reproduces the price of a given American put option with strike $K$ and maturity $T$.

In figure 3, we test the approximation ((13), (14)) of the exercise boundary with discrete dividends, where $D_n$ $=$ 4% each year.

- **Local volatility model.** We now check the validity of (20) in the case of a local volatility model, where $\sigma(T,K)$ has been constructed using a (synthetic) European implied volatility. The results are reported in figure 2 for $T$ $=$ 2 years (a test for maturity $T$ $=$ 1 year can be found in De Marco & Henry-Labordère (2016)). As before, the result is almost exact when using the first-order approximation. When using (17) or (19), we extend the approximation $\tau$ $\mapsto$ $\frac{1}{\sqrt{\mathbb{E} T}}(\tau)$ with a constant beyond the value of $\tau$ (if any), where this function loses its monotonicity (figure 2(b)).

Figure 4 shows the results in the case of discrete dividends.

- **SABR model.** Results in the case of the SABR model are displayed in De Marco & Henry-Labordère (2016, figure 4).

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