A Cornish-Fisher Expansion Using Six Cumulants

Abramowitz and Stegun (1972) provide a higher order expansion of the cumulative distribution inverse. Using their notations, we have

$$\tilde{T}_\alpha \sim \bar{\mu}_1 + \sqrt{\bar{\mu}_2}\delta(\alpha),$$

where

$$\begin{aligned}
\delta(\alpha) = x &+ [\gamma_1 h_1(x)] \\
&+ \left[\gamma_2 h_2(x) + \gamma_1^2 h_{11}(x)\right] \\
&+ \left[\gamma_3 h_3(x) + \gamma_1\gamma_2 h_{12}(x) + \gamma_1^3 h_{111}(x)\right] \\
&+ \left[\gamma_4 h_4(x) + \gamma_2^2 h_{22}(x) + \gamma_1\gamma_3 h_{13}(x) + \gamma_1^2\gamma_2 h_{112}(x) + \gamma_1^4 h_{1111}(x)\right],
\end{aligned}$$

$$\begin{aligned}
x &= \Phi^{-1}(\alpha) \\
h_1(x) &= \frac{1}{6}He_2(x) \\
h_2(x) &= \frac{1}{24}He_3(x) \\
h_{11}(x) &= -\frac{1}{36}\left[2He_3(x) + He_1(x)\right] \\
h_3(x) &= \frac{1}{120}He_4(x) \\
h_{12}(x) &= -\frac{1}{24}\left[He_4(x) + He_2(x)\right] \\
h_{111}(x) &= \frac{1}{324}\left[12He_4(x) + 19He_2(x)\right] \\
h_4(x) &= \frac{1}{720}He_5(x) \\
h_{22}(x) &= -\frac{1}{384}\left[3He_5(x) + 6He_3(x) + 2He_1(x)\right] \\
h_{13}(x) &= -\frac{1}{180}\left[2He_5(x) + 3He_3(x)\right] \\
h_{112}(x) &= \frac{1}{288}\left[14He_5(x) + 37He_3(x) + 8He_1(x)\right] \\
h_{1111}(x) &= -\frac{1}{7776}\left[252He_5(x) + 832He_3(x) + 227He_1(x)\right], \\
\gamma_{r-2} &= \frac{\kappa_r(\tilde{T}_n)}{\bar{\mu}_2^{r/2}}, r = 3, 4, ..., \\
He_n(x) &= 2^{-n/2}H_n(\frac{x}{\sqrt{2}}).
\end{aligned}$$

$H_n(.)$ is the Hermite polynomial of degree n. It can be computed using the recursive relation

$$\begin{aligned}
H_n(x) &= 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \forall n \geq 2, \\
H_1(x) &= 2x, \\
H_0(x) &= 1.
\end{aligned}$$

$\kappa_r(\tilde{T})$ are the cumulants of the empirical aggregate loss distribution. Recall that

$$\begin{aligned} \tilde{T} &= \sum_{i=1}^{n} f^i \tilde{x}_i, \\ f^i &\sim \frac{\lambda}{n}. \end{aligned}$$

This implies that

$$\begin{aligned} \kappa_r(\tilde{T}) &= \sum_{i=1}^{n} \kappa_r(\tilde{x}_i) \\ &= \kappa'_r \sum_{i=1}^{n} \tilde{x}_i^r, \end{aligned}$$

where

$$\kappa'_r = \kappa_r(f^i).$$

The latter results is based on the additive property of cumulants. Denote the r^{th} central moments of f^i by μ_r^i

$$\mu_r^i = E\left[(f^i - \frac{\lambda}{n})^r\right]$$

Therefore,

$$\begin{aligned} \kappa'_3 &= \mu_3^i \\ \kappa'_4 &= \mu_4^i - 3\mu_2^{i2} \\ \kappa'_5 &= \mu_5^i - 10\mu_2^i\mu_3^i \\ \kappa'_6 &= \mu_6^i - 15\mu_2^i\mu_4^i - 10\mu_3^{i2} + 30\mu_2^{i3} \end{aligned}$$

DasGupta (1998) provides a formula to compute the central moments of a Poisson distribution. We obtain

$$\mu_r^i = \sum_{k=0}^{r} a_{k,r}(\frac{\lambda}{n})^k,$$

where

$$a_{k,r} = \sum_{i=0}^{k} (-1)^i \binom{r}{i} S_2(r-i, k-i).$$

$S_2(.,.)$ is the Stirling number of the second kind, and is defined as

$$S_2(r,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^r,$$

and $S_2(0,0) = 1$.