MUREX™

LOGICAL SPACE™

The problems with generally used interpolation spaces

In a world increasingly focused on effective enterprise-level risk management, there are notable discrepancies in volatility management techniques. Murex proposes a cross-asset interpolation space with potentially significant risk management impacts

Volatility, as a crucial parameter for derivatives trading, is common to all asset classes. It is not, however, represented consistently, which precludes comprehensive cross-asset volatility transparency and management. As with many things, 'the devil is in the detail' and something as seemingly simple as the choice of interpolation method can have a significant impact. Most 'time interpolations' in 'spaces' such as 'strike' or 'moneyness' are relatively meaningless as they ignore price dispersion over time implicit in volatility. A commonly used compromise is to adjust moneyness using 'at-the-money' volatility (ATMV) interpolated at the square root of time. Using only ATMV ignores, however, the reality of smile. 'Delta space' interpolation is consistent through time but is challenged by multiple definitions of delta. Even ignoring this challenge, the computational intensity of translating from smiled delta space back to strike or moneyness space results in less logical but simpler methods prevailing. Other methods, such as interpolating the parameters of a model, for example, SABR[1], between two dates are satisfactory but are exclusively linked to the given parametric method.

'Probability space'

Probability is a fundamental concept in all options pricing. 'Breeden-Litzenberger[2]' gives an analytical method of generating a risk-neutral density function from any differentiable smile, regardless of underlying asset and smile construction. We propose interpolating in probability space, which like delta is consistent over time but avoids the questions of delta definition. To interpolate the volatility $\sigma_T(p)$ for a given probability p from the corresponding volatilities at the surrounding expirations, we use the standard method of linearly interpolating $\sigma_T(p)^2 T$ between $\sigma_1(p)^2 T_1$ and $\sigma_2(p)^2 T_2$.

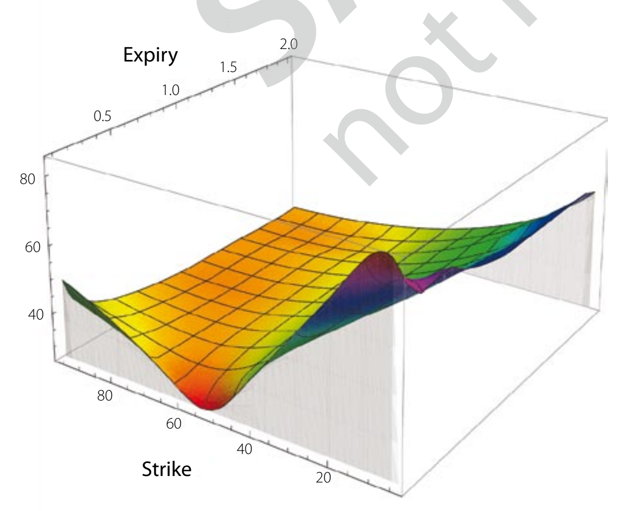
Benefits of probablility space

Breeden-Litzenberger has been widely accepted since publication in 1978. Additionally, it does not depend on any specific smile interpolation method. As such, we believe probability space interpolation could be adopted by the broadest practitioner community affording homogeneity across asset classes. As noted earlier, most markets do not work in a truly time-comparable space and, as such, volatility surfaces lack transparency. The time homogeneity of probability space is evident in figures 1 and 2, comparing Deutsche Bank equity volatility in 'strike space' (figure 1) with probability space (figure 2).

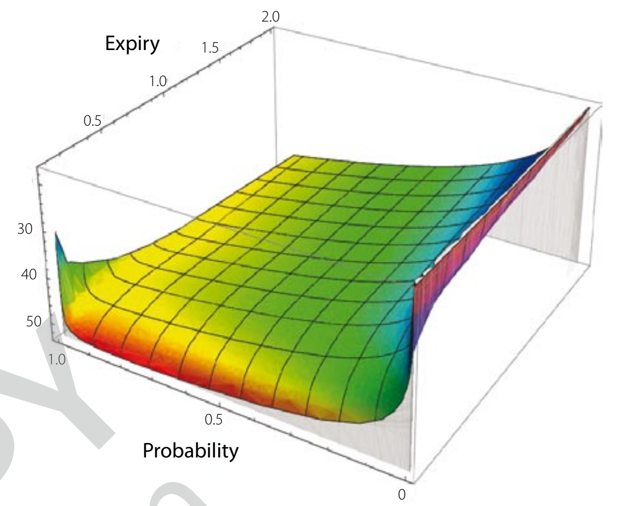
Describing a surface in probability space allows creation of comparable 'observables', for example, call volatility at 25% and 75% probability points relative to ATMV. This allows immediate observation of whether skewness or leptokurtosis increases over time, across assets and asset classes. This has potential to provide significant tractability of volatility surfaces for enterprise-wide risk management. We also note that this interpolation method is also compatible with 'day weighting' or 'adjusted time'.

We believe the benefits of managing volatilities within a Logical Space™ are significant and, as such – while integrating the functionality within Murex's systems – are publishing all details of our research for

1 Deutsche Bank volatility surface, December 2009: in strike space



2 Deutsche Bank volatility surface, December 2009: in logical space



free utilisation or extension by any interested parties. In the interests of brevity, only high-level details are provided in this article, but the more detailed white paper *Logical Space*™ has been submitted for publication and is freely available on request from **logical@murex.com**

Sanity checking

Probability space interpolation does not guarantee arbitrage-free smiles, but interpolated smiles can be expected to be as arbitrage-free as the smiles they depend upon. This allows users to choose between the robustness of parametrically generated smiles and the simplicity of geometrically generated smiles. A range of sanity checks are recommended in the white paper *Logical Space*™.

Interpolation to avoid computational intensity

Probability space encounters the same computational difficulties as translating delta space back into strike or moneyness space. It is usual when translating to solely focus on the interpolated smile; the curves from which the interpolation was made are ignored. We propose interpolating characteristics of these adjacent smiles to approximate strike or moneyness space.

The objective is to produce an approximate moneyness curve for the intermediate maturity that closely satisfies the following version (equation 1) of Breeden-Litzenberger[2]:

$$p = 1 - N\left(\frac{-M - \frac{\sigma_T(p)^2 T}{2}}{\sigma_T(p)\sqrt{T}}\right) + \sqrt{T}N'\left(\frac{-M - \frac{\sigma_T(p)^2 T}{2}}{\sigma_T(p)\sqrt{T}}\right)\frac{\frac{d\sigma_T(p)}{dp}}{D} \quad (1)$$

with $M = k_{approx}(p)$ and $D = \frac{dk_{approx.}(p)}{dp}$, where $p = \mathrm{P}(S_T \leq F_T e^{k_{approx}(p)})$ represents the probability that, at maturity, the spot will be at or below the strike corresponding to the moneyness $k_{approx}(p)$.

We do not prescribe any specific method but have tested several and devised a straightforward analytical 'moneyness/slope-mixing' method that, given ongoing testing, appears accurate.

'Moneyness interpolation'

This consists of interpolating the moneyness $k(p)$ for a given probability p at intermediate expiration T from the moneyness' $k_1(p)$ and $k_2(p)$ for this probability p at the surrounding expirations T_1 and T_2. This method is not always satisfactory when checking the accuracy by regenerating the density function using Breeden-Litzenberger

The 'slope' method

This method can be regarded in some respects as an error-predictor/corrector modification of moneyness interpolation. For a given probability P, this method consists in setting $k_{slope}(p) = M_{implied}$, where $M_{implied}$ is the value of M that corresponds, via equation (1), to:

$$D = \frac{dk_{mon.\mathrm{int.}}(p)}{dp} \quad (2)$$

Our numerical tests indicate that the slope method is not generally better (or worse) than the underlying moneyness interpolation method; the moneyness adjustment $k_{slope}(p) - k_{mon.\mathrm{int.}}$ when passing from $k_{mon.\mathrm{int.}}(p)$ to $k_{slope}(p)$ tends to be too large.

The moneyness/slope-mixing method

Because the slope method generally 'overshoots' when correcting the 'moneyness method', the errors are generally partially offsetting. The moneyness/slope-mixing method simply consists of combining the moneyness interpolation method and the slope method:

$$k_{average}(p) = \frac{k_{mon.\mathrm{int.}}(p) + k_{slope}(p)}{2} \quad (3)$$

With this method, the moneyness adjustment step from $k_{mon.\mathrm{int.}}(p)$ to $k_{average}(p)$ is half the size of the step from $k_{mon.\mathrm{int.}}$ to k_{slope}.

We have tested other methods and, in the context of our tests, the moneyness/slope-mixing method delivered the most consistently accurate results. This does not preclude further improvements.

Logical Space™ was first presented at the Frankfurt Mathfinance Conference on March 16, 2010. Murex gratefully acknowledges Professor Dr Uwe Wystup's review of Logical Space™.

[1] *SABR (Stochastic Alpha, Beta, Rho) refers to Hagan, PS, D Kumar, AS Lesniewski & DE Woodward (2002), Managing smile risk,* Wilmott Magazine, *September, pp. 84–108.*

[2] *'Breeden-Litzenberger' refers to Breeden D & R Litzenberger (1978), Prices of state-contingent claims implicit in option prices,* Journal of Business *51, pp. 621–652.*

The classical Breeden-Litzenberger'equation relates the risk-neutral probability density of the spot at maturity T to the second derivative, with respect to the strike K, of the price of the corresponding call option $C_T(K)$. Integrating this equation with respect to the strike yields the following expression for the cumulative probability p (K) of the spot to be at or below K at maturity:

$$p(K) = 1 + e^{rT}\frac{dC_T(K)}{dK}$$

Expressing the call price in this equation in terms of the 'Black-Scholes'-Scholes formula and passing to (log) moneyness k leads to equation (1). We use the term Breeden-Litzenberger in a broader sense, which includes the form of equation (1).