The implied volatility surface in the presence of dividends

Standard Bank quantitative analyst Roelof Sheppard shows how absolute and proportional dividend payments give rise to arbitrage constraints on the implied volatility surface

Introduction

In South African markets, European call and put options trade liquidly at certain strikes and maturities on the TOPI40, an index of the top-40 shares based on market capitalisation. In general, it is assumed that dividends on indexes pay frequently, hence it is appropriate to make use of a dividend yield, and that implied volatilities for non-market options can be interpolated from liquid options. In this article, we show that these assumptions can introduce arbitrage around dividend dates. We also provide specific arbitrage formulae, which relate implied volatilities on either side of dividend dates.

It is reasonable to assume that different institutions will have different dividend forecasts. Furthermore, these forecasts may be modelled as discrete dividends, either proportional to the stock price or an absolute cash amount. In this article we will show how absolute and proportional dividend payments give rise to different arbitrage constraints on the implied volatility surface. Consequently, two institutions who agree on all the market option prices may imply quite different volatilities and prices for non-standard options.

Arbitrage constraints on implied volatility

The cash price and volatility of an option are related as usual via:

$$V_{t,T}(K) = e^{-r(T-t)} Black(f_{t,T}, \sigma_{T,K}, K, T) \qquad (1)$$

where $V_{t,T}(K)$ is the value of a European option at time t with a strike K, maturity T and implied volatility $\sigma_{T,K}$. This option is written on an underlying with a forward price given by $f_{t,T}$. The interest rate is denoted by r.

To obtain a relationship for the implied volatility over a dividend date we consider a European call option with a dividend paid at the maturity of the option. We will discuss discrete absolute dividends and discrete proportional dividends separately.

Absolute dividends

Making use of the fact that the spot price will decrease by the amount of the dividend on the dividend date, we obtain the following formula:

$$S_{t_D^+} = S_{t_D^-} - D \qquad (2)$$

where t_D^- and t_D^+ denote the times immediately before and after the dividend payment of size D.

The forward price just after the dividend payment is given by:

$$f_{t,t_D^+} = S_t e^{r(T-t)} - D$$

By substituting this into equation (1) we obtain the value of an option maturing on t_D^+:

$$V_{t,t_D^+}(K) = e^{-r(T-t)} Black(S_t e^{r(T-t)} - D, \sigma_{t_D^+,K}, K, T) \qquad (3)$$

The value of the option maturing immediately after the dividend date is the same as the value of the option maturing just before a dividend date with an adjusted strike. To see this, consider the following trades:

☐ **Trade 1:** Buy a call option maturing on t_D^- with strike $K+D$. If this is in the money we buy the stock for $K+D$. Since we are then long the stock when the dividend pays, we will receive D in cash, so we will have effectively paid K for the stock.

☐ **Trade 2:** Buy a call option maturing on t_D^+ with strike K. Equation (2) shows that, when trade 1 is in the money so is trade 2 and both will give the holder the stock for a price of K. Similarly and trivially they both have the same zero value when they are out of the money.

Alternatively we can see this via:

$$\begin{aligned} V_{t_D^+,t_D^+}(K) &= \max(S_{t_D^+} - K, 0) \\ &= \max((S_{t_D^-} - D) - K, 0) \\ &= \max(S_{t_D^-} - (K + D), 0) \\ &= V_{t_D^-,t_D^-}(K + D) \end{aligned}$$

Combining the arbitrage argument above with the fact the value of an option maturing just before a dividend date is given by:

$$\begin{aligned} &V_{t,t_D^+}(K+D) \\ &= e^{-r(T-t)}Black(S_t e^{r(T-t)}, \sigma_{t_D^-,K+D}, K+D, T) \end{aligned}$$

we obtain the following implicit relationship for implied volatilities over dividend dates:

$$\begin{aligned} &Black(S_t e^{r(T-t)} - D, \sigma_{t_D^+,K}, K, T) \\ &= Black(S_t e^{r(T-t)}, \sigma_{t_D^-,K+D}, K+D, T) \end{aligned} \qquad (4)$$

Given market prices of options maturing just before a dividend date, equation (4) can be used to obtain arbitrage-free implied volatilities after dividend dates. A numerical root solver such as Newton's method can be used to obtain the required implied volatility $\sigma_{t_D^+,K}$.

Proportional dividends

If we assume now that the dividend is proportional to the stock price just before the dividend, i.e., $D=\alpha S_{t_D^-}$ equation (2) becomes:

$$S_{t_D^+} = S_{t_D^-}(1-\alpha) \qquad (5)$$

The value of the option maturing just after the dividend is now given by:

$$V_{t,t_D^+}(K) = e^{-r(T-t)}Black(S_{t_D^-}(1-\alpha)e^{r(T-t)}, \sigma_{t_D^+,K}, K, T)$$

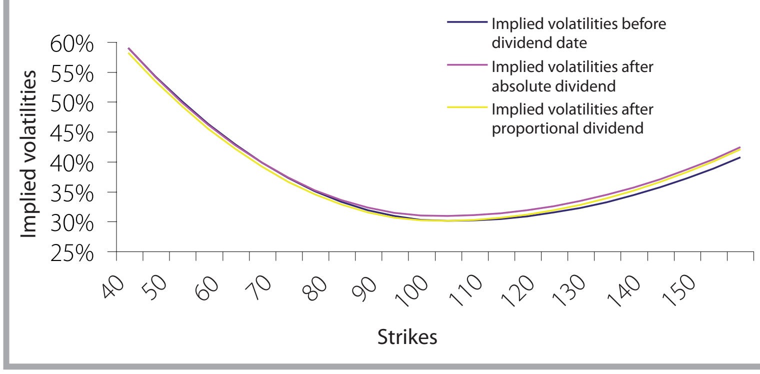
In this case we have the following relationship:

$$\begin{aligned} &V_{t_D^+,t_D^+}(K) = \max(S_{t_D^+} - K, 0) \\ &= \max(S_{t_D^-}(1-\alpha) - K, 0) \\ &= (1-\alpha)\max\left(S_{t_D^-} - K/(1-\alpha), 0\right) \\ &= (1-\alpha)V_{t_D^-,t_D^-}\left(K/(1-\alpha)\right) \end{aligned}$$

From this equation we obtain the following arbitrage relationship for implied volatilities over a dividend date:

$$\begin{aligned} &Black(S_{t_D^-}(1-\alpha)e^{r(T-t)}, \sigma_{t_D^+,K}, K, T) \\ &= (1-\alpha)Black\left(S_{t_D^-}e^{r(T-t)}, \sigma_{t_D^-,K/(1-\alpha)}, K/(1-\alpha), T\right) \\ &= Black\left(S_{t_D^-}(1-\alpha)e^{r(T-t)}, \sigma_{t_D^-,K/(1-\alpha)}, K, T\right) \end{aligned}$$

1 Comparison of implied volatilities over a dividend date



This equation implies the following arbitrage relationship:

$$\sigma_{t_D^+,K} = \sigma_{t_D^-,K/(1-\alpha)} \qquad (6)$$

Once again this equation can be used to obtain arbitrage-free implied volatilities after dividend dates from market prices of options maturing just before a dividend date.

An example

Consider a case when the underlying spot price is 100, the agreed forward price is 105 and a dividend is paid in one year's time at the maturity of the option. If we assume that the interest rate is 7%, it can be shown that this market data implies an absolute dividend of D=2.25 and a proportional dividend of α=2.1%.

Given a skew just before a dividend date, figure 1 shows which implied volatilities after a dividend date should be used to avoid arbitrage.

From figure 1, it is clear that absolute and proportional dividends will enforce different arbitrage constraints on the implied volatility surface.

Conclusion

These discontinuities in volatility over dividend dates are not seen on the market implied surface since options are not available immediately before and after the dividend dates. These arbitrage constraints are significant when:

- Option traders transact over-the-counter options and interpolate to obtain volatilities.
- Models are calibrated to the whole implied volatility surface to price exotic options.

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