A Willow tree construction

Now, we give details on the constructions of the willow tree for three popular stochastic processes, such as the geometric Brownian motion (GBM), Merton's jump-diffusion (MJD) (Merton, 1976) and Heston's stochastic volatility (HSV) (Heston, 1993) models.

A.1 Geometric Brownian motion

Assume the underlying asset price S_t under the $\mathbb{Q}$ measure are governed by the stochastic differential equation (SDE)

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{A.24}$$

where r is the risk-free interest rate, W_t is the standard Brownian motion, σ is the constant volatility. The value of S_t at t can be estimated as

$$S_t = S_0 \ \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}. \tag{A.25}$$

Thus, the discrete asset price S_i^n at t_n can be estimated as

$$S_i^n = S_0 \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t_n + \sigma\sqrt{t_n} z_i \right\}, \quad \text{for } i = 1, ..., m. \tag{A.26}$$

where z_i is the discrete value chosen from the standard normal distribution following the sample strategy in Xu et al. (2013). According to Xu et al. (2013), a sequence of $\{(z_i,\ \hat{q}_i)\}$, $i = 1, 2, \cdots, m$, is generated to approximate the standard normal distribution, where z_i is some discrete value of the standard normal distribution and $\hat{q}_i$ is the corresponding probability of z_i.

The transition probability from S_i^n to S_j^{n+1}, p_{ij}^n, can then be estimated as (Lu and Xu, 2017)

$$p_{ij}^n = P(Y_j^{n+1}|Y_i^n) = \int_{c_j^{n+1}}^{c_{j+1}^{n+1}} f(y|Y_i^n)\, dy, \quad \text{for } i, j = 1, ..., m,$$

where $Y_i^n \equiv \sqrt{t_n} z_i$, $c_j^{n+1} = (Y_j^{n+1} + Y_{j-1}^{n+1})/2$, $c_{j+1}^{n+1} = (Y_{j+1}^{n+1} + Y_j^{n+1})/2$, $c_1^{n+1} = -\infty$, $c_{m+1}^{n+1} = +\infty$ for $j = 1, 2, ..., m$, and $f(y|Y_i^n)$ is the conditional probability density function for a normally distributed random variable y at t_{n+1}, given Y_i^n, i.e.,

$$f(y|Y_i^n) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left\{ -\frac{(y - Y_i^n)^2}{2\Delta t} \right\},$$

for $n = 1, ..., N-1$. The transition probability from S_0 to S_j^1, q_j, can be determined by

$$q_j = P(Y_j^1|Y^0) = \int_{c_j^n}^{c_{j+1}^1} f(y)\, dy,$$

where $f(y) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left\{ -\frac{y^2}{2\Delta t} \right\}$.

A.2 Merton's jump-diffusion model

Assume the underlying asset price S_t follows a jump-diffusion process (Merton, 1976) as

$$\frac{dS_t}{S_t} = (r - \tilde{\lambda}\bar{k})dt + \sigma dW_t + [Y_t - 1]\, dN_t,$$

where r is the constant risk-free interest rate, W_t is the standard $\mathbb{Q}$-Brownian motion, $\bar{k} = \mathbb{E}[Y_t - 1]$, $\ln Y_t \sim N(\alpha_J, \sigma_J^2)$, and N_t follows the Poisson process with constant intensity $\tilde{\lambda}$. The first four moments of the log-return of S_t, $X_t = \ln(S_t/S_0)$ can be computed analytically (Ballotta and Kyriakou, 2015) as

$$\begin{aligned}
\text{Mean} &= [r - \tfrac{\sigma^2}{2} - \tilde{\lambda}(e^{\alpha_J + \sigma_J^2/2} - 1) + \tilde{\lambda}\alpha_J]t \\
\text{Variance} &= (\sigma^2 + \tilde{\lambda}\alpha_J^2 + \tilde{\lambda}\sigma_J^2)t \\
\text{Skewness} &= \frac{\tilde{\lambda}(\alpha_J^3 + 3\alpha_J\sigma_J^2)}{\sqrt{t}(\sigma^2 + \tilde{\lambda}\alpha_J^2 + \tilde{\lambda}\sigma_J^2)^{3/2}} \\
\text{Kurtosis} &= \frac{\tilde{\lambda}(\alpha_J^4 + 6\alpha_J^2\sigma_J^2 + 3\sigma_J^4)}{t(\sigma^2 + \tilde{\lambda}\alpha_J^2 + \tilde{\lambda}\sigma_J^2)^2} + 3.
\end{aligned} \tag{A.27}$$

The Johnson curve transformation (Johnson, 1949) transforms a standard normal variable into an arbitrary random variable via matching the first four moments. The nodes are set to be

$$X_i^n = \varepsilon g^{-1}\left(\frac{z_i - \gamma}{\delta}\right) + \nu, \tag{A.28}$$

where the parameters γ, δ, ν and ε can be determined by the algorithm proposed in Hill and Holder (1976), z_i are the discrete values of the standard normal distribution and the function $g^{-1}(u)$ is defined by

$$g^{-1}(u) = \begin{cases} e^u & \text{for the lognormal family,} \\ \frac{e^u - e^{-u}}{2} & \text{for the unbounded family,} \\ \frac{1}{1+e^{-u}} & \text{for the bounded family,} \\ u & \text{for the normal family.} \end{cases} \tag{A.29}$$

The m possible log-returns X_i^n, $i = 1, 2, \cdots, m$, are selected to match the first four moments of X_{t_n} by the Johnson curve transformation. The key in sampling X_i^n is to select $\{z_i\}$ from the standard normal distribution. The corresponding underlying asset prices S_i^n on the willow tree can then be calculated as $S_i^n = S_0 \exp(X_i^n)$.

The transition probability p_{ij}^n from X_i^n to X_j^{n+1} can be estimated by (Xu and Yin, 2014)

$$p_{ij}^n = P(A < X_j^{n+1} < B | X_i^n) = \int_{C_j^{n+1}}^{C_{j+1}^{n+1}} \sum_{l=0}^{\infty} \frac{e^{-\tilde{\lambda}\Delta t}(\tilde{\lambda}\Delta t)^l}{l!} \frac{1}{\sqrt{2\pi}\sigma_l} \exp\left[-\frac{(x-\mu_l)^2}{2\sigma_l^2}\right] dx, \tag{A.30}$$

where $C_j^{n+1} = (X_{j-1}^{n+1} + X_j^{n+1})/2$, $C_{j+1}^{n+1} = (X_{j+1}^{n+1} + X_j^{n+1})/2$, $C_1^{n+1} = -\infty$, $C_{m+1}^{n+1} = +\infty$, $\mu_l = X_i^n + (r - \tilde{\lambda}\bar{k} - \sigma^2/2)\Delta t + l\alpha_J$ and $\sigma_l^2 = \sigma^2\Delta t + l\sigma_J^2$.

A.3 Heston's stochastic volatility model

Assume the underlying asset price S_t follows a Heston stochastic volatility model (Heston, 1993)

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1, \\ dv_t = \eta(\varpi - v_t)dt + \sigma_v \sqrt{v_t} dW_t^2, \end{cases} \tag{A.31}$$

where r is the risk-free interest rate, ϖ is the long-term mean of variance, η is a mean-reverting speed parameter of the variance, σ_v is the so-called volatility of volatility. The two Wiener processes dW_t^1 and dW_t^2 are assumed to be correlated with a constant correlation coefficient ρ, that is $\mathbb{E}^{\mathbb{Q}}[dW_t^1 dW_t^2] = \rho dt$. To ensure the variance is always positive, the Feller condition must be satisfied, i.e., $2\eta\varpi \geq \sigma_v^2$.

When the Feller condition is satisfied, the details of a two-dimensional willow tree construction for S_t and v_t can be referred in Ma et al. (2020b). In summary, the willow tree consists of the tree nodes, that are pairs of possible values of S_t and v_t, $(S_i^n,\ v_{i_1}^n)$, at time $t_n = n \cdot \Delta t$, where $i = 1, 2, \cdots, m$ and $i_1 = \lfloor i/m_x \rfloor + 1$, m_x is the number of possible values of S_t at t_n given a possible value of v_t and $\lfloor a \rfloor$ returns the largest integer less than a, and the transition probability between $(S_i^n,\ v_{i_1}^n)$ at t_n and $(S_j^{n+1},\ v_{j_1}^{n+1})$ defined as p_{ij}^n.

When the Feller condition is violated, the variance in the Heston model could be negative in our willow tree or Monte Carlo simulation. In this paper, we adopt the adaptation in Cozma and Reisinger (2020) to set the negative variance to be zero. For example, at $t_n \equiv n\Delta t$, one of the variance on the willow tree, $v_{i_1}^n$, is negative, i.e., $v_{i_1}^n < 0$. We first set it to be zero, i.e., $v_{i_1}^n = 0$. Then, the transition probability from tree node at t_n, $(S_i^n, v_{i_1}^n)$, to the tree nodes, $(S_j^{n+1}, v_{j_1}^{n+1})$, at t_{n+1}, p_{ij}^n, can be determined as

$$p_{ij}^n = \begin{cases} 1 & \text{if } \ln S_i^n + r\Delta t \in [\frac{\ln S_j^{n+1} + \ln S_{j-1}^{n+1}}{2},\ \frac{\ln S_j^{n+1} + \ln S_{j+1}^{n+1}}{2}] \\ & \quad \text{and} \quad \eta\varpi\Delta t \in [\frac{v_{j_1}^{n+1} + v_{j_1-1}^{n+1}}{2},\ \frac{v_{j_1}^{n+1} + v_{j_1+1}^{n+1}}{2}] \\ 0 & \text{otherwise} \end{cases},$$

given $v_{i_1}^n = 0$ for $j = 1, 2, \cdots, m$ and $j_1 = \lfloor j/m_x \rfloor + 1$.

B Proof of Proposition 1

Proof. Equation (3.6) can be written as

$$\begin{aligned} \exp\left(-\frac{\theta}{1-R}(n+1)\Delta t\right) &= \mathbb{E}\left[\exp\left(-\sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \alpha(t, V) dt\right)\right] \\ &\approx \mathbb{E}\left[\exp\left(-\sum_{k=0}^{n} \alpha^k \Delta t\right)\right]. \end{aligned} \tag{B.32}$$

where $\alpha^k \triangleq \alpha(t_k, V^k)$. Dividing both sides of the equation (B.32) by $\exp(-\frac{\theta}{1-R}(n+1)\Delta t)$, we have

$$\begin{aligned} 1 &= \mathbb{E}\left[\exp\left(-\sum_{k=0}^{n}\alpha^k\Delta t + \frac{\theta}{1-R}(n+1)\Delta t\right)\right] \\ &= \mathbb{E}\left[\prod_{k=0}^{n}\exp\left(-\alpha^k\Delta t + \frac{\theta}{1-R}\Delta t\right)\right] \\ &= \mathbb{E}\left[\prod_{k=0}^{n}\eta^k\right], \end{aligned} \tag{B.33}$$

where $\eta^n \triangleq \exp\left(-\alpha^n\Delta t + \frac{\theta}{1-R}\Delta t\right)$.

For $n = 0$, according to (B.33), we have

$$\mathbb{E}[\eta^0] = \exp\left(-\alpha^0\Delta t + \frac{\theta}{1-R}\Delta t\right) = 1,$$

i.e., $\alpha^0 = \exp(a^0 + bV^0) = \frac{\theta}{1-R}$.

For $n = 1$, we have

$$\mathbb{E}[\eta^0\eta^1] = \mathbb{E}[\eta^1] = \sum_{i=1}^{m} q_i\eta_i^1 = 1.$$

Let $w_i^1 = q_i$, and we obtain equation (3.8).

For $n > 1$, on the one hand, $\sum\limits_{i=1}^{m} w_i^n\eta_i^n$ can be written as, based on the definition of w_i^n

$$\begin{aligned} \sum_{i=1}^{m} w_i^n\eta_i^n &= \sum_{i=1}^{m}\eta_i^n[\sum_{j_{n-1}=1}^{m} p_{j_{n-1}i}^{n-1}\eta_{j_{n-1}}^{n-1}w_{j_{n-1}}^{n-1}] \\ &= \sum_{i=1}^{m}\eta_i^n[\sum_{j_{n-1}=1}^{m} p_{j_{n-1}i}^{n-1}\eta_{j_{n-1}}^{n-1}[\sum_{j_{n-2}=1}^{m} p_{j_{n-2}j_{n-1}}^{n-2}\eta_{j_{n-2}}^{n-2}w_{j_{n-2}}^{n-2}]] \\ &= \sum_{i=1}^{m}\eta_i^n[\sum_{j_{n-1}=1}^{m} p_{j_{n-1}i}^{n-1}\eta_{j_{n-1}}^{n-1}\cdots[\sum_{j_1=1}^{m} p_{j_1j_2}^1\eta_{j_1}^1 w_{j_1}^1]]. \end{aligned} \tag{B.34}$$

On the other hand, due to $\eta^0 = 1$, $\mathbb{E}[\prod\limits_{k=0}^{n}\eta^k]$ can be expressed discretely in the willow tree framework as the sum of $\prod\limits_{k=1}^{n}\eta_{j_k}^k$ with probability $q_{j_1}\prod\limits_{k=1}^{n-1}p_{j_kj_{k+1}}^k$ for each $j_1, j_2, ..., j_n = 1, 2, \cdots, m$, i.e.,

$$\begin{aligned} \mathbb{E}[\prod_{k=1}^{n}\eta^k] &= \sum_{j_n=1}^{m}\cdots\sum_{j_1=1}^{m}(\prod_{k=1}^{n}\eta_{j_k}^k)(q_{j_1}\prod_{k=1}^{n-1}p_{j_kj_{k+1}}^k) \\ &= \sum_{j_n=1}^{m}\eta_{j_n}^m[\sum_{j_{n-1}=1}^{m} p_{j_{n-1}j_n}^{n-1}\eta_{j_{n-1}}^{n-1}\cdots[\sum_{j_1=1}^{m} p_{j_1j_2}^1\eta_{j_1}^1 q_{j_1}]]. \end{aligned} \tag{B.35}$$

Then, according to (B.34), (B.35) and (B.33), we have

$$\sum_{i=1}^{m} w_i^n \eta_i^n = \mathbb{E}[\prod_{k=0}^{n} \eta^k] = 1.$$

Thus, we have proved equation (3.8).

□

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