

APPENDIX A. FINITE VERSUS
INFINITE-ACTIVITY-LÉVY-PROCESSES

Appendix Finite Infinite

Proposition A.1. *From Rémillard (2013), let L be a Lévy process with characteristics (a, b, k) , where k is the Lévy measure defined on \mathbb{R} such that $k(\{0\}) = 0$ and $\int_{\mathbb{R} \setminus 0} (1 \wedge |x|^2) k(dx) \leq \infty$. In a finite time interval, the number of jumps of Lévy process can be finite or infinite, according as $k(\mathbb{R}) < \infty$ or $k(\mathbb{R}) = \infty$.*

Proposition A.2. *From Rémillard (2013), let L be a Lévy process with characteristics (a, b, k) , where k is the Lévy measure defined on \mathbb{R} such that $k(\{0\}) = 0$ and $\int_{\mathbb{R} \setminus 0} (1 \wedge |x|^2) k(dx) \leq \infty$. A Lévy process has jumps of finite variation if and only if $b = 0$ and $\int_{|x| < 1} |x| k(dx) < \infty$.*

APPENDIX B. TRANSITION TABLES

Appendix Transition1

B.1. Transition tables - Merton (1976). The transition parameters $T_{k,i}^\nu$, for $\nu \in \{0, 1, 2\}$, $k \in \{1, \dots, p\}$, and $i \in \{0, \dots, p\}$ are

$$T_{k,i}^\nu = \sum_{n=0}^{\infty} \mathbb{Q}(N_{\Delta t} = n) \eta_k^\nu(n) e^{c(n)^2/2} \left[\Phi(c_{k,i+1}(n) - c(n)) - \Phi(c_{k,i}(n) - c(n)) \right],$$

where $N_{\Delta t}$ is the number of jumps over $[t_m, t_{m+1}]$, $c(n) = \nu \sigma_n \sqrt{\Delta t}$, and

$$\begin{aligned} \mathbb{Q}(N_{\Delta t} = n) &= e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!}, \\ \sigma_n^2 &= \sigma^2 + \frac{n}{\Delta t} \delta^2, \\ \eta_k(n) &= a_k e^{(r - \bar{d} - \lambda \kappa - \sigma_n^2/2) \Delta t + n(\gamma + \delta^2/2)}, \\ c_{k,i}(n) &= \frac{\log(a_i/a_k) - (r - \bar{d} - \lambda \kappa - \sigma_n^2/2) \Delta t - n(\gamma + \delta^2/2)}{\sigma_n}, \end{aligned}$$

and $\Phi(\cdot)$ is the standard normal distribution function.

Appendix Transition2

B.2. Transition tables - Kou (2002). The transition parameters $T_{k,i}^\nu$, for $\nu \in \{0, 1, 2\}$, $k \in \{1, \dots, p\}$, and $i \in \{0, \dots, p\}$ are

$$\begin{aligned} T_{k,i}^0 &= \Upsilon(\mu_0, \sigma, \lambda, p_1, \eta_1, \eta_2, x_{i+1}, \Delta t) - \Upsilon(\mu_0, \sigma, \lambda, p_1, \eta_1, \eta_2, x_i, \Delta t), \\ T_{k,i}^1 &= \rho^{-1} a_k [\Upsilon(\mu_1, \sigma, \tilde{\lambda}, \tilde{p}_1, \tilde{\eta}_1, \tilde{\eta}_2, x_{i+1}, \Delta t) - \Upsilon(\mu_1, \sigma, \tilde{\lambda}, \tilde{p}_1, \tilde{\eta}_1, \tilde{\eta}_2, x_i, \Delta t)], \\ T_{k,i}^2 &= b \rho^{-2} a_k^2 [\Upsilon(\mu_2, \bar{\sigma}, \bar{\lambda}, \bar{p}_1, \bar{\eta}_1, \bar{\eta}_2, \bar{x}_{i+1}, \Delta t) - \Upsilon(\mu_2, 2\sigma, \bar{\lambda}, \bar{p}_1, \bar{\eta}_1, \bar{\eta}_2, \bar{x}_i, \Delta t)], \end{aligned}$$

where $\mu_0 = r - \frac{1}{2}\sigma^2 - \lambda\kappa$, $x_i = \log(a_i/a_k)$, $\rho = \exp(-(r - \bar{d})\Delta t)$, $\mu_1 = r + \frac{1}{2}\sigma^2 - \lambda\kappa$, $\tilde{\lambda} = \lambda(1 + \kappa)$, $\tilde{p}_1 = p\eta_1/(1 + \kappa)(\eta_1 - 1)$, $\tilde{\eta}_1 = \eta_1 - 1$, $\tilde{\eta}_2 = \eta_2 + 1$, $\bar{\sigma} = 2\sigma$, $\bar{\kappa} = p_1(\eta_1/2\bar{\eta}_1) + (1 - p_1)(\eta_2/2\bar{\eta}_2) - 1$, $\mu_2 = 2r + \frac{1}{2}\bar{\sigma}^2 - \lambda\bar{\kappa}$, $\bar{\lambda} = \lambda(1 + \bar{\kappa})$, $\bar{\eta}_1 = \eta_1/2 - 1$, $\bar{\eta}_2 = \eta_2/2 + 1$, $b = \exp(\sigma^2 + \lambda(\bar{\kappa} - 2\kappa)\Delta t)$, and $\bar{x}_i = x_i - \log(b)$. The function $\Upsilon(\cdot)$ is defined by

$$\begin{aligned} \Upsilon(\mu, \sigma, \lambda, \eta_1, \eta_2, p_1, x_i, \Delta t) &= \frac{e^{(\sigma\eta_1)^2\Delta t/2}}{\sigma\sqrt{2\pi\Delta t}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^n P_{n,k} \left(\sigma\sqrt{\Delta t}\eta_1 \right)^k \\ &\times I_{k-1} \left(x_i - \mu\Delta t; -\eta_1, -\frac{1}{\sigma\sqrt{\Delta t}}, -\sigma\eta_1\sqrt{\Delta t} \right) \\ &+ \frac{e^{(\sigma\eta_2)^2\Delta t/2}}{\sigma\sqrt{2\pi\Delta t}} \sum_{n=1}^{\infty} \pi_n \sum_{k=1}^n Q_{n,k} \left(\sigma\sqrt{\Delta t}\eta_2 \right)^k \\ &\times I_{k-1} \left(x_i - \mu\Delta t; \eta_2, \frac{1}{\sigma\sqrt{\Delta t}}, -\sigma\eta_2\sqrt{\Delta t} \right) \\ &+ \pi_0 \Phi \left(-\frac{x_i - \mu\Delta t}{\sigma\sqrt{\Delta t}} \right), \end{aligned}$$

and by

$$\begin{aligned} P_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{i-k} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-i} p_1^i p_2^{n-i}, \\ Q_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k} \binom{n}{i} \cdot \left(\frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-i} \left(\frac{\eta_2}{\eta_1 + \eta_2} \right)^{i-k} p_1^{n-i} p_2^i, \\ I_n(c; \alpha, \beta, \delta) &= \int_c^{\infty} e^{\alpha x} Hh_n(\beta x - \delta) dx, \end{aligned}$$

for arbitrary constants $\alpha, c, \beta \in \mathbb{R}$, and $n \in \mathbb{N}$.

B.3. Transition tables – Variance Gamma - Madan et al. (1998).
From Madan et al. (1998), we define the degenerate hypergeometric

function of two variables $\Psi(a, b, \gamma)$ in terms of the modified Bessel function of the second kind $K(\cdot)$ as

$$\begin{aligned} \Psi(a, b, \gamma) &= \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^\gamma}{\sqrt{(2\pi)}\Gamma(\gamma)\gamma} K_{\gamma+\frac{1}{2}}(c) \times \\ &\quad \Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)) - \\ &\quad \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{1+\gamma}}{\sqrt{(2\pi)}\Gamma(\gamma)(1+\gamma)} K_{\gamma-\frac{1}{2}}(c) \times \\ &\quad \Phi(1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)) + \\ &\quad \text{sign}(a) \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^\gamma}{\sqrt{(2\pi)}\Gamma(\gamma)\gamma} K_{\gamma-\frac{1}{2}}(c) \times \\ &\quad \Phi(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)), \end{aligned}$$

where $c = |a| \sqrt{2+b^2}$, $u = b/\sqrt{2+b^2}$, and where the degenerate hypergeometric function of two variables Φ has the integral representation

$$\Phi(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} e^{uy} du.$$

Let $x = \frac{1+u}{2}$, $\lambda = 2 \text{sign}(a)c$, so that $c = |\lambda|/2$, and set

$$\begin{aligned} \Psi_1(x, \lambda, \gamma) &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^\gamma}{2\sqrt{\pi}\Gamma(\gamma)\gamma} K_{\gamma+\frac{1}{2}}(c) \Phi(\gamma, 1-\gamma, 1+\gamma; x, -\lambda x) \\ &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^\gamma}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(c) \int_0^1 z^{\gamma-1} (1-zx)^{\gamma-1} e^{-\lambda zx} dz \\ \Psi_2(x, \lambda, \gamma) &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma+1}}{\sqrt{\pi}\Gamma(\gamma)(\gamma+1)} K_{\gamma-\frac{1}{2}}(c) \Phi(1+\gamma, 1-\gamma, 2+\gamma; x, -\lambda x) \\ &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^{\gamma+1}}{\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(c) \int_0^1 z^\gamma (1-zx)^{\gamma-1} e^{-\lambda zx} dz \\ \Psi_3(x, \lambda, \gamma) &= \frac{|\lambda|^{\gamma+\frac{1}{2}} e^{\lambda/2} x^\gamma}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(c) \int_0^1 z^{\gamma-1} (1-zx)^{\gamma-1} e^{-\lambda zx} dz. \end{aligned}$$

If $\lambda > 0$, let $t = \lambda xz$, set

$$\Psi_1(x, \lambda, \gamma) = \frac{\sqrt{\lambda}e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(\lambda/2) \int_0^{\lambda x} t^{\gamma-1} \left(1 - \frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt,$$

and set $I_1(x, \lambda, \gamma) = \int_0^{\lambda x} t^{\gamma-1} \left(1 - \frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt$. Integrating by parts,

$$\begin{aligned} I_1(x, \lambda, \gamma) &= \frac{(\lambda x)^\gamma}{\gamma} (1-x)^{\gamma-1} e^{-\lambda x} + \frac{1}{\gamma} \int_0^{\lambda x} t^\gamma \left(1 - \frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt \\ &\quad + \frac{\gamma-1}{\lambda\gamma} \int_0^{\lambda x} t^\gamma \left(1 - \frac{t}{\lambda}\right)^{\gamma-2} e^{-t} dt. \end{aligned}$$

Set $h_1(x, \lambda, \gamma) = \int_0^{\lambda x} t^\gamma \left(1 - \frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt$ and $h_2(x, \lambda, \gamma) = \int_0^{\lambda x} t^\gamma \left(1 - \frac{t}{\lambda}\right)^{\gamma-2} e^{-t} dt$, where h_1 and h_2 are evaluated by Gauss-Legendre quadrature. Then

$$\begin{aligned} \Psi_2(x, \lambda, \gamma) &= \frac{e^{\lambda/2}}{\sqrt{\pi\lambda}} K_{\gamma-\frac{1}{2}}(\lambda/2) \int_0^{\lambda x} t^\gamma \left(1 - \frac{t}{\lambda}\right)^{\gamma-1} e^{-t} dt \\ &= \frac{e^{\lambda/2}}{\sqrt{\pi\lambda}} K_{\gamma-\frac{1}{2}}(\lambda/2) h_1(x, \lambda, \gamma), \end{aligned}$$

and $\Psi_3(x, \lambda, \gamma) = \frac{\sqrt{\lambda}e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(\lambda/2) I_1(x, \lambda, \gamma)$, hence

$$\Psi(x, \lambda, \gamma) = \Psi_1(x, \lambda, \gamma) - \text{sign}(a) \Psi_2(x, \lambda, \gamma) + \text{sign}(a) \Psi_3(x, \lambda, \gamma).$$

If $\lambda < 0$, let $t = -\lambda xz$, let

$$\Psi_1(x, \lambda, \gamma) = \frac{\sqrt{-\lambda}e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma+\frac{1}{2}}(-\lambda/2) \int_0^{-\lambda x} t^{\gamma-1} \left(1 + \frac{t}{\lambda}\right)^{\gamma-1} e^t dt,$$

and set $I_2(x, \lambda, \gamma) = \int_0^{-\lambda x} t^{\gamma-1} \left(1 + \frac{t}{\lambda}\right)^{\gamma-1} e^t dt$. Integrating by parts,

$$\begin{aligned} I_2(x, \lambda, \gamma) &= \frac{(-\lambda x)^\gamma}{\gamma} (1-x)^{\gamma-1} e^{-\lambda x} - \frac{1}{\gamma} \int_0^{-\lambda x} t^\gamma \left(1 + \frac{t}{\lambda}\right)^{\gamma-1} e^t dt \\ &\quad - \frac{\gamma-1}{\lambda\gamma} \int_0^{-\lambda x} t^\gamma \left(1 + \frac{t}{\lambda}\right)^{\gamma-2} e^t dt. \end{aligned}$$

Set $h_3(x, \lambda, \gamma) = \int_0^{-\lambda x} t^\gamma \left(1 + \frac{t}{\lambda}\right)^{\gamma-1} e^t dt$ and $h_4(x, \lambda, \gamma) = \int_0^{-\lambda x} t^\gamma \left(1 + \frac{t}{\lambda}\right)^{\gamma-2} e^t dt$, where h_3 and h_4 are evaluated by Gauss-Legendre quadrature. Then

$$\begin{aligned} \Psi_2(x, \lambda, \gamma) &= \frac{\sqrt{(-\lambda)}e^{\lambda/2}}{\lambda\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(-\lambda/2) h_3(x, \lambda, \gamma), \\ \Psi_3(x, \lambda, \gamma) &= \frac{\sqrt{(-\lambda)}e^{\lambda/2}}{2\sqrt{\pi}\Gamma(\gamma)} K_{\gamma-\frac{1}{2}}(-\lambda/2) I_2(x, \lambda, \gamma). \end{aligned}$$

As a result,

$$\Psi(x, \lambda, \gamma) = \Psi_1(x, \lambda, \gamma) - \text{sign}(a) \Psi_2(x, \lambda, \gamma) + \text{sign}(a) \Psi_3(x, \lambda, \gamma).$$

The transition parameters $T_{k,i}^j$, for $j \in \{0, 1, 2\}$, $k \in \{1, \dots, p\}$, and $i \in \{0, \dots, p\}$ are

$$\begin{aligned} T_{k,i}^0 &= \Psi(x_0, \lambda_i^{(0)}, dt/\nu) - \Psi(x_0, \lambda_{i+1}^{(0)}, dt/\nu), \\ T_{k,i}^1 &= \rho^{-1} a_k \left[\Psi(x_1, \lambda_i^{(1)}, dt/\nu) - \Psi(x_1, \lambda_{i+1}^{(1)}, dt/\nu) \right], \\ T_{k,i}^2 &= e^{\eta_2} \rho^{-2} a_k^2 \left[\Psi(x_2, \lambda_i^{(2)}, dt/\nu) - \Psi(x_2, \lambda_{i+1}^{(2)}, dt/\nu) \right], \end{aligned}$$

where $\rho = \exp\{-(r-q)dt\}$, $x_0 = \frac{1+u_0}{2}$, $u_0 = \frac{b_0}{\sqrt{2+b_0^2}}$, $b_0 = \alpha \sqrt{\frac{\nu}{1-\xi_2}}$, $\xi_2 = \frac{\nu\alpha^2}{2}$, $\alpha = \zeta s$, with $\zeta = \frac{\theta}{\sigma^2}$ and $s = \frac{\sigma}{\sqrt{1+(\frac{\theta}{\sigma})^2 \frac{\nu}{2}}}$. Thus $\lambda_i^{(0)} = 2 \operatorname{sign}(a_0)c_0$, with $c_0 = |a_0| \sqrt{2+b_0^2}$, $a_0 = d_i \sqrt{\frac{1-\xi_2}{\nu}}$, $\xi_1 = \frac{\nu(\alpha+s)^2}{2}$, and

$$d_i^{(0)} = \frac{1}{s} \left[\ln \left(\frac{a_k}{a_i} \right) + rdt + \frac{dt}{\nu} \ln \left(\frac{1-\xi_1}{1-\xi_2} \right) \right].$$

Next, $x_1 = \frac{1+u_1}{2}$, $u_1 = \frac{b_1}{\sqrt{2+b_1^2}}$, $b_1 = (\alpha+s) \sqrt{\frac{\nu}{1-\xi_1}}$, $\lambda_i^{(1)} = 2 \operatorname{sign}(a_1)c_1$, with $c_1 = |a_1| \sqrt{2+b_1^2}$, $a_1 = d_i^{(0)} \sqrt{\frac{1-\xi_1}{\nu}}$.

Further, let $r_2 = 2r$, $\sigma_2 = 2\sigma$, $\theta_2 = 2\theta$, $q_2 = 2q$, $\alpha_2 = \zeta_2 s_2$, with $\zeta_2 = \frac{\theta_2}{\sigma_2^2}$ and $s_2 = \frac{\sigma_2}{\sqrt{1+(\frac{\theta_2}{\sigma_2})^2 \frac{\nu}{2}}}$. Then $x_2 = \frac{1+u_2}{2}$, $u_2 = \frac{b_2}{\sqrt{2+b_2^2}}$, $b_2 = (\alpha_2 + s_2) \sqrt{\frac{\nu}{1-\xi_1^{(2)}}}$, with $\xi_1^{(2)} = \frac{\nu(\alpha_2+s_2)^2}{2}$. Thus, $\lambda_i^{(2)} = 2 \operatorname{sign}(a_2)c_2$, with $c_2 = |a_2| \sqrt{2+b_2^2}$, $a_2 = d_i^{(1)} \sqrt{\frac{1-\xi_1^{(2)}}{\nu}}$, $\xi_2^{(2)} = \frac{\nu\alpha_2^2}{2}$ and

$$d_i^{(2)} = \frac{1}{s_2} \left[2 \ln \left(\frac{a_k}{a_i} \right) + r_2 dt + \eta_2 + \frac{dt}{\nu} \ln \left(\frac{1-\xi_1^{(2)}}{1-\xi_2^{(2)}} \right) \right],$$

where $\eta_2 = 2w_1 - w_2$, $w_1 = \frac{dt}{\nu} \ln \left(\frac{1-\xi_1}{1-\xi_2} \right)$, and $w_2 = \frac{dt}{\nu} \ln \left(\frac{1-\xi_1^{(2)}}{1-\xi_2^{(2)}} \right)$.

Thus, for $j \geq 3$

$$T_{k,i}^j = e^{\eta_j} \rho^{-j} a_k^j \left[\Psi(x_j, \lambda_i^{(j)}, dt/\nu) - \Psi(x_j, \lambda_{i+1}^{(j)}, dt/\nu) \right],$$

where $r_j = jr$, $\sigma_j = j\sigma$, $\theta_j = j\theta$, $q_j = jq$, $\alpha_j = \zeta_j s_j$, with $\zeta_j = \frac{\theta_j}{\sigma_j^2}$ and $s_j = \frac{\sigma_j}{\sqrt{1+(\frac{\theta_j}{\sigma_j})^2 \frac{\nu}{2}}}$. Hence, $x_j = \frac{1+u_j}{2}$, $u_j = \frac{b_j}{\sqrt{2+b_j^2}}$, $b_j =$

$(\alpha_j + s_j) \sqrt{\frac{\nu}{1-\xi_1^{(j)}}}$, with $\xi_1^{(j)} = \frac{\nu(\alpha_j + s_j)^2}{2}$, $\xi_2^{(j)} = \frac{\nu\alpha_j^2}{2}$, and $\lambda_i^{(j)} = 2 \operatorname{sign}(a_j) c_j$,

with $c_j = |a_j| \sqrt{2 + b_j^2}$, $a_j = d_i^{(j)} \sqrt{\frac{1-\xi_2^{(j)}}{\nu}}$, and

$$d_i^{(j)} = \frac{1}{s_j} \left[j \ln \left(\frac{a_k}{a_i} \right) + r_j dt + \eta_j + \frac{dt}{\nu} \ln \left(\frac{1 - \xi_1^{(j)}}{1 - \xi_2^{(j)}} \right) \right],$$

where $\eta_j = jw_1 - w_j$, $w_1 = \frac{dt}{\nu} \ln \left(\frac{1-\xi_1}{1-\xi_2} \right)$, and $w_j = \frac{dt}{\nu} \ln \left(\frac{1-\xi_1^{(j)}}{1-\xi_2^{(j)}} \right)$.

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