Black basket analytics for mid-curves and spread options

Alexandre Antonov proposes a new way to unify two marginal distributions with a large number of parameters, permitting mid-curve or spread options with multiple strikes to be calibrated. The method is based on a basket of lognormal processes (called a Black basket) having a fast analytical formulation and attractive simplicity. Moreover, the application of the Black basket method can be extended to full term structure models: Black basket LMM

Mid-curve and spread options are popular semi-vanillas whose pricing is closely related via the 'copularisation' of two fixed marginal distributions of two standard interest rates, R_1 and R_2, calibrated to their markets. The main question is how to construct their joint distribution such that their general 'basket' option prices $\mathbb{E}[(w_1 R_1 + w_2 R_2 - K)^+]$ for different strikes K will yield market information. There is a large literature devoted to various copulas of these distributions: see Andersen & Piterbarg (2010) for a detailed review. Standard copulas are parameterised with a very limited number of parameters. For example, the Gaussian copula has only one parameter, a correlation, which can eventually be fixed, and calibrated to one strike. Quite often, this is not sufficient – the market can contain information on multiple strikes. We also mention the so-called power Gaussian copula (see Andersen & Piterbarg 2010), which formally has two extra parameters (the powers). However, they do not permit full three-dimensional manipulation: lower powers make the effective correlation weaker and unable to match market prices. Another drawback of the copula approach is its numerical realisation. Of course, we can reduce the resulting two-dimensional integral to a one-dimensional integral (see, for example, Borst 2014) including a bivariate cumulative distribution function. However, the latter can still be slow and noisy, especially for calibration of the numerous mid-curve/spread options observed on the market.

An alternative approach is the explicit identification of rates as part of some multi-dimensional process. For example, Andersen & Piterbarg (2010) consider each rate being driven with the Heston process with some stochastic volatility. Each Heston model is calibrated to the standard option market, and the joint distribution freedom can be associated with cross-correlations between two rates and their stochastic volatilities. One problem is that the spread option price is not available analytically unless we employ a two-dimensional Laplace transform (Antonov & Arneguy 2009). Hagan *et al* (2020) calculated an analytical expression for normal stochastic alpha–beta–rho underlyings, which could potentially help. In this paper, however, we are looking for a general solution with an extendible flexibility for both underlyings and correlations, as well as accurate analytics. For this, we apply this multi-dimensional process approach to an analytically solvable basket of lognormal processes that is flexible enough to be calibrated to complicated underlying distributions and has a large cross-correlation freedom to be calibrated to multiple generalised basket options. Below we will refer to such a basket as the 'Black basket' (see, for example, Carmona & Durrleman 2006; Pellegrino 2016; Shimko 1994).

Our main technical contribution is the construction of joint distributions of two rates, R_1 and R_2, both represented as Black baskets with several lognormal terms. Namely, cross-correlations between the lognormal components of R_1 and R_2 give a sufficient degree of freedom to calibrate their spread options to multiple strikes (naturally, such a joint distribution corresponds to some multi-parameter copula). We also come up with an efficient analytical pricing algorithm and numerically confirm the calibration capabilities of the Black basket as well as its parameters and hedging stability.

Finally, we apply the Black basket logic for a term-structure modelling that generalises the London Interbank Offered Rate (Libor) market model (LMM), making it flexible yet allowing a simple and efficient analytical approximation.

Setup and measure changes

Denote a general forward swap price and corresponding rate as:

$$S(t; T_1, T_2) = P(t, T_1) - P(t, T_2) - K A(t; T_1, T_2)$$

and:

$$R(t; T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{A(t; T_1, T_2)}$$

where K is the strike, $P(t, T)$ is the zero-coupon bond price and:

$$A(t; T_1, T_2) = \sum_n \delta t_n P(t, t_n)$$

is an annuity with payment dates $t_1, t_2, \ldots$ for $\delta t_n = t_n - t_{n-1}$. In practice, the annuity numerator and denominator, representing different interest rates, can be more complicated. Here, for brevity, we represent them in the classical way using the same zero bond rate. We adopt the notation $S_i(t) = S(t; T_0, T_i)$, $R_i(t) = R(t; T_0, T_i)$ and $A_i(t) = A(t; T_0, T_i)$ for standard swaps, swap rates and annuities, respectively, where T_0 is a fixing date and T_i is an end date.

Let us now define two main instruments of interest – a mid-curve swaption and a spread option – and reduce them to calculable expressions involving a standard rate distribution in their martingale measures.

■ **Mid-curve.** A mid-curve swaption is an option to enter to a swap $S(T_0; T_1, T_2)$ with start T_1 and end T_2 at exercise date T_0. Denoting mid-curve objects with a subscript 'mc', ie, $S_{\mathrm{mc}}(t) = S(t; T_1, T_2)$, $R_{\mathrm{mc}}(t) = R(t; T_1, T_2)$ and $A_{\mathrm{mc}}(t) = A(t; T_1, T_2)$, we can write the mid-curve swaption price in the annuity A_{mc} measure as:

$$V_{\mathrm{mc}}(0) = A_{\mathrm{mc}}(0) \mathbb{E}^{A_{\mathrm{mc}}}[(R_{\mathrm{mc}}(T_0) - K)^+] \tag{1}$$

under which the mid-curve forward rate process $R_{\mathrm{mc}}(t)$ is a martingale. We further represent the option price as:

$$V_{\mathrm{mc}}(0) = A_{\mathrm{mc}}(0) \mathbb{E}^{A_{\mathrm{mc}}}\left[\left(\hat{R}_2(T_0)\frac{A_2(0)}{A_{\mathrm{mc}}(0)} - \hat{R}_1(T_0)\frac{A_1(0)}{A_{\mathrm{mc}}(0)} - K\right)^+\right] \tag{2}$$

such that the ‘hat’ rates:

$$\hat{R}_i(t) = R_i(t)\frac{A_i(t)}{A_{\mathrm{mc}}(t)}\frac{A_{\mathrm{mc}}(0)}{A_i(0)} \tag{3}$$

are martingales in the $\mathbb{E}^{A_{\mathrm{mc}}}[\cdot]$ measure, ie, $\mathbb{E}^{A_{\mathrm{mc}}}[\hat{R}_i(T_0)] = R_i(0)$. Moreover, we can calculate the hat rates’ distributions (encoded in terms of options) given a distribution of the standard rates in their martingale measure together with a measure change process (A_i-martingale):

$$\hat{M}_i(t) = \frac{A_{\mathrm{mc}}(t)}{A_i(t)}\frac{A_i(0)}{A_{\mathrm{mc}}(0)}$$

as follows:

$$\mathbb{E}^{A_{\mathrm{mc}}}[(\hat{R}_i(T_0) - K)^+] = \mathbb{E}^{A_i}[(R_i(T_0) - K\hat{M}_i(T_0))^+] \tag{4}$$

Without a term-structure model we do not possess any accurate information about rate–annuity interdependence. In principle, say, the first martingale, $\hat{M}_1$, is supposed to be some nonlinear function of both rates R_1 and R_2. Such an approach was followed in Feldman (2020). Here, we simply approximate the martingale by its conditional expectation given the corresponding rate:

$$\hat{M}_i(T_0) \simeq \mathbb{E}^{A_i}[\hat{M}_i(T_0) \mid R_i(T_0)] \tag{5}$$

such that the expectation in the right-hand side of (4), being dominated by the rate $R_i(T_0)$, remains close to its exact value after the approximation (5). Then, we can safely assume that the conditional expectation $\mathbb{E}^{A_i}[\hat{M}_i(T_0) \mid R_i(T_0)]$ is a linear function of the rate:

$$\hat{M}_i(t) \simeq \mathbb{E}^{A_i}[\hat{M}_i(T_0) \mid R_i(T_0)] \simeq 1 + \hat{\lambda}_i(R_i(t) - R_i(0)) \tag{6}$$

A linear coefficient $\hat{\lambda}_i$ can be calculated as above by supposing the main mode of the yield curve is a parallel shift. Finally, the measure-change relation (4) can be approximated as:

$$\mathbb{E}^{A_{\mathrm{mc}}}[(\hat{R}_i(T_0) - K)^+] \simeq \mathbb{E}^{A_i}[(R_i(T_0)(1 - \hat{\lambda}_i K) - K(1 - \hat{\lambda}_i R_i(0)))^+] \tag{7}$$

Note that the approximate measure change process (6) can, in principle, be negative, but this would happen only for extreme rates. Moreover, we can check that the second derivative over the strike of the right-hand side of (7) is always positive, which makes the approximation arbitrage-free.

Let us justify here our approximation choices. As current interest rates are very low and their volatility is not large, the linear dependence (6) makes perfect sense.[1] A flat yield assumption does not contradict the *raison d’être* of the mid-curve/spread option, ie, as a measure of independent movements of different parts of the yield curve. Indeed, we approximate measure change martingales in the one-rate expectations (4) but not the underlying rates in the final mid-curve/spread formulas. Moreover, the stochasticity carried over from the rate $\hat{\lambda}_i(R_i(t) - R_i(0))$ via the measure change is very mild with respect to the first, deterministic, part. See Antonov (2020) and references therein for more examples.

■ **Spread option.** A spread option gives, at exercise date T_0, the right to pay a spread between two rates versus a strike, $R_2(T_0) - R_1(T_0) - K$, at a payment date T_p. Using standard arguments, we express it as a price in the T_p-forward measure:

$$V_{\mathrm{sp}}(0) = P(0, T_p)\mathbb{E}^{T_p}[(R_2(T_0) - R_1(T_0) - K)^+] \tag{8}$$

For the spread option pricing (8) we should be able to calculate rates’ distributions in the T_p-forward measure given their distributions in the martingale annuity measure. For both interest rates R_1 and R_2, we have the following measure-change relation:

$$\mathbb{E}^{T_p}[f(R_i(T_0))] = \mathbb{E}^{A_i}[M_i(T_0)f(R_i(T_0))] \tag{9}$$

for any function $f(\cdot)$ and martingales:

$$M_i(t) = \frac{P(t, T_p)}{A_i(t)}\frac{A_i(0)}{P(0, T_p)} \tag{10}$$

for $i = 1, 2$. Following the above logical relation (6), we approximate each martingale as a linear function of its own rate, $M_i(t) \simeq 1 + \lambda_i(R_i(t) - R_i(0))$. A linear coefficient λ_i can be calculated as in the mid-curve case by a parallel shift in the yield curve. Then, the measure-change relation (9) can be approximated as:

$$\mathbb{E}^{T_p}[f(R_i(T_0))] \simeq \mathbb{E}^{A_i}[(1 + \lambda_i(R_i(t) - R_i(0)))f(R_i(T_0))] \tag{11}$$

Below we will see how we can use this relation to get the rates in the T_p-measure.

In the next section, we introduce our main technical tool: a basket of lognormal (Black) processes.

Black basket rate representation

The Black basket form is a rate representation as a sum of lognormal (Black) processes. It is a generalisation of the displaced diffusion process and has enough degrees of freedom to be calibrated to a smile. Specifically, a Black basket form is the following function of vector Brownian motions $W_i(t)$:

$$R(t) = R_0 + \sum_i \alpha_i(\mathrm{e}^{\sigma_i W_i(t) - \frac{1}{2}\sigma_i^2 t} - 1) \tag{12}$$

with the following parameters: volatilities σ_i, weights α_i and correlations $\rho_{ii'}$:

$$\rho_{ii'}\,\mathrm{d}t = \mathbb{E}[\mathrm{d}W_i(t)\,\mathrm{d}W_{i'}(t)] \tag{13}$$

For the single rate distribution we can use 100% correlated Brownian motions (BMs), but for the main Black basket usage – mid-curves or spread options – a decorrelation of BMs, ie, $\rho_{ij} \neq 1$, brings extra freedom in calibration.

An option price for the Black basket rate can be solved analytically (see the appendix in Antonov (2020)). Importantly, a product of two Black basket rates and their linear combination can also be presented in Black basket form. This means that rates in different measures as well as their spreads still possess the Black basket structure and thus can be solved analytically.

The correlated Black basket has linear asymptotics in a normal implied volatility; in other words, it is close to a lognormal process for large deviations. For a general decorrelated Black basket, the asymptotics are sublinear.

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[1] *For very large maturities, when the movement in rates can be significant, we can represent the conditional expectation* $\mathbb{E}^{A_i}[\hat{M}_i(T_0) \mid R_i(T_0)]$ *as a nonlinear function of* $R_i(T_0)$*, denoted by* $\hat{m}_i(R_i(T_0))$*. Then, the option price (4) can be calculated using the Carr–Madan static replication formula applied to* $\mathbb{E}^{A_i}[(R_i(T_0) - K\hat{m}_i(R_i(T_0)))^+]$*, where the natural distribution of the rate* $R_i(T_0)$ *is known. In the case of extreme maturities and volatilities, when the nonlinear effects dominate, we can use a term-structure model (see the section below called ‘The Black basket term structure model: generalising LMM’).*

To approximate the Black basket at-the-money (ATM) volatility, we expand the exponents in (12) to obtain a Gaussian approximation:

$$R(t) \simeq R_0 + G(t) \quad \text{for } G(t) = \sum_i \alpha_i \sigma_i W_i(t) \tag{14}$$

Then, the ATM volatility can be simply approximated by the volatility of the Gaussian process $G(t)$, $\sigma^2_{\text{ATM}} = \sum_{i,j} \alpha_i \sigma_i \alpha_j \sigma_j \rho_{ij}$. Approximate skew and curvature can be obtained for some special cases, including the case of two exponents below.

■ **Simplest Black basket examples.** Consider a case of two exponents – as the first generalisation of the displaced diffusion – with 'opposite' volatilities:

$$R(t) = R_0 + \alpha_1(\mathrm{e}^{\sigma W(t) - \frac{1}{2}\sigma^2 t} - 1) + \alpha_2(\mathrm{e}^{-\sigma W(t) - \frac{1}{2}\sigma^2 t} - 1) \tag{15}$$

The biggest smile effect is when $\alpha_1 \sim -\alpha_2$, in particular, for the hyperbolic sine form:

$$R(t) \simeq R_0 + 2\alpha \sinh(\sigma W(t)) \tag{16}$$

Our workhorse will be a general two-exponential form:

$$R(t) = R_0 + \alpha_1(\mathrm{e}^{\sigma_1 W_1(t) - \frac{1}{2}\sigma_1^2 t} - 1) + \alpha_2(\mathrm{e}^{-\sigma_2 W_2(t) - \frac{1}{2}\sigma_2^2 t} - 1) \tag{17}$$

where we have left two different volatilities and decorrelated the BMs. As already mentioned, this decorrelation gives us extra freedom in the calibration of mid-curves and spread options.

■ **Small time smile expansion.** For the correlated two-exponential case (15) we can devise an efficient analytical small time expansion useful for both understanding the smile and an initial calibration estimate. A normal implied volatility reads:

$$\hat{\sigma}(K) = \sigma d(1 + \tfrac{1}{2} y q + \tfrac{1}{2} y^2(\tfrac{1}{3} - \tfrac{1}{2} q^2)) + O(y^3)$$

where:

$$y = \frac{K - R_0}{d} \quad \text{and} \quad q = \frac{s}{d}$$

for the α values' sum, $s = \alpha_1 + \alpha_2$, and difference, $d = \alpha_1 - \alpha_2$ (see Antonov (2020) for details). We see that the ATM smile level is equal to the approximate normal volatility, a skew is proportional to q (a zero-skew case corresponds to a symmetric case (16)), and the curvature is $\frac{1}{3} - \frac{1}{2} q^2$.

Rate calibration

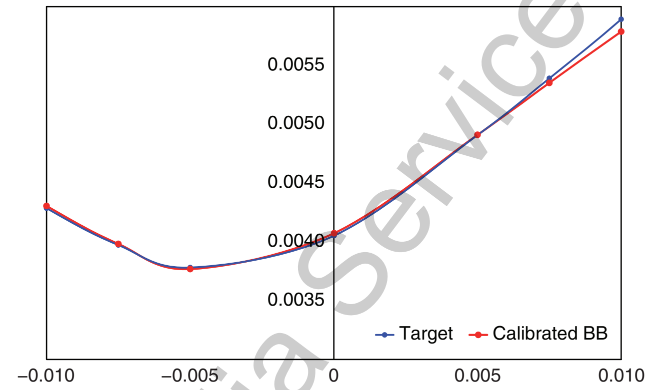
In this section we will consider the calibration of rates represented by the Black basket (12). This form appears to be both sufficiently flexible to fit a market smile and stable with respect to a linear combination and a multiplication. The latter permits us to compute the rate characteristics in different measures in order to address mid-curve and spread options.

■ **Calibration to swaptions: examples.** First, we will start with examples of rate calibration to European swaptions. Rate parameters in (12) are calibrated using a numerical solver with the estimate provided by the analytical approximation in the previous subsection.

In all the experiments in this paper, we use an observation date of April 3, 2020, and treat euro instruments. Figure 1 gives a calibration accuracy of several swaptions with 1Y expiry with relatively pronounced smiles. It shows both target (market) normal implied volatility and the resulting calibrated one corresponding to the Black basket model with parameters presented in table A.

Now, we discuss rates' representation in the mid-curve annuity measure and the forward annuity measure given their calibrated Black basket form in the martingale measure.

1 Black basket model calibration to the 1Y5Y European swaption



Normal implied volatility (vertical axis) as a function of $K - F_0$ (horizontal axis), ie, a strike K position with respect to the forward F_0. Axis units are absolute

A. Calibrated Black basket parameters corresponding to the 1Y5Y case (17)

F_0	σ_1	σ_2	α_1	α_2	ρ
–0.0019	0.5132	0.5132	0.00624	–0.00441	0

■ **Rates in the mid-curve annuity measure.** In the 'Set up and measure changes' section we saw that the hat rates $\hat{R}_i$ (3) underlie the mid-curve price (2). Their distributions in the mid-curve annuity measure $\mathbb{E}^{A_{\text{mc}}}[\cdot]$ are defined by (7), where the right-hand side depends on distributions of the standard rates R_i already calibrated in their martingale measures (12). We represent the rates $\hat{R}_i$ in the Black basket form:

$$\hat{R}(t) = R_0 + \sum_i \alpha_i (\mathrm{e}^{\hat{\sigma}_i W_i^{A_{\text{mc}}}(t) - \frac{1}{2}\hat{\sigma}_i^2 t} - 1) \tag{18}$$

and calibrate the parameters $\{\hat{\sigma}_i\}$ to option prices (7) keeping the same weights α_i as in the standard rate.[2] A numerical solver will rapidly converge if we set the initial estimated volatilities to those of the standard rate.

■ **Rates in the forward measure.** As we have seen, the spread option price (8) depends on the rates' distribution in the forward measure. To obtain it we will use the linear approximation (11). The values of interest – a constant maturity swap (CMS) forward $\mathbb{E}^{T_p}[R(T_0)]$ and CMS option prices $\mathbb{E}^{T_p}[(R(T_0) - K)^+]$ – can be calculated analytically given the pre-calibrated distribution of the rate in its martingale measures (12). Indeed, the CMS forward depends on the rate variance in the martingale measure:

$$\mathbb{E}^{T_p}[R(T_0)] = R(0) + \lambda \mathbb{E}^A[(R(T_0) - R(0))^2]$$

and can be easily evaluated analytically. For options in the forward measure:

$$\begin{aligned}&\mathbb{E}^{T_p}[(R(T_0) - K)^+] \\ &\quad = \mathbb{E}^A[(R(T_0) - K)^+] + \lambda \mathbb{E}^A[(R(T_0) - K)^+ (R(T_0) - R(0))]\end{aligned}$$

both terms can be expressed as a sum of options on new Black basket rates (see Antonov 2020). Thus, given the Black basket rate representation in the

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[2] *Of course, we can include both the volatilities and the weights in the calibration parameters. However, our choice was inspired by the form of the hat rate* $\hat{R}_i = R_i / \hat{M}_i$*, where the process* $M_i^{-1}(t)$*, being an* A_i*-martingale, can be approximated as a lognormal process. Thus, the ratio* $R_i / \hat{M}_i$ *will keep approximately the same weights but have a modified volatility part.*

martingale measure (12) and the measure-change recipe (11) we can calculate the CMS forward and CMS options. Conversely, if, say, the CMS forward is available on the market (via CMS swaps), it is possible to calibrate the linear coefficient λ to it.

In what follows, while working with the forward measure, we will modify the rate amplitudes $\alpha \to \alpha'$ and the initial value $R_0 \to R'_0$:

$$R'(t) = R'_0 + \sum_i \alpha'_i (\mathrm{e}^{\sigma_i W'_i(t) - \frac{1}{2}\sigma_i^2 t} - 1) \tag{19}$$

keeping the same volatilities (we denote the forward measure values by a prime). One reason for this is that a measure change modifies the averages of BMs that re-translate to amplitudes, leaving the volatilities unchanged. We naturally require R'_0 to be equal to the CMS forward, and the expectations $\mathbb{E}^{T_p}[(R'(T_0) - K)^+]$ to fit the CMS option prices for certain strikes. Then, fixing R'_0, we calibrate numerically the amplitudes α' to CMS options. Their set of strikes is chosen to match the number of variable parameters.

Black basket model for mid-curves and spread options

We can represent a mid-curve or spread option price as:

$$\mathcal{O} = \mathbb{E}[(\delta_1 r_1(T_0) + \delta_2 r_2(T_0) - k)^+] \tag{20}$$

for two zero-mean Black basket rates:

$$\left.\begin{aligned} r_1(t) &= \sum_i \alpha_i^{(1)} (\mathrm{e}^{\sigma_i^{(1)} W_i^{(1)}(t) - \frac{1}{2}(\sigma_i^{(1)})^2 t} - 1) \\ r_2(t) &= \sum_j \alpha_j^{(2)} (\mathrm{e}^{\sigma_j^{(2)} W_j^{(2)}(t) - \frac{1}{2}(\sigma_j^{(2)})^2 t} - 1) \end{aligned}\right\} \tag{21}$$

For the mid-curve option case (2), the coefficients' δ values are defined as a ratio of annuities – $\delta_1 = -A_1(0)/A_{\mathrm{mc}}(0)$ and $\delta_2 = A_2(0)/A_{\mathrm{mc}}(0)$ – and can potentially have different scales: one of them can be dominating. The generalised strike k reads $k = K - R_{\mathrm{mc}}(0)$, while the effective rates r_i are the hat rates (18) minus the average, $r_i = \hat{R}_i - R_i(0)$.

For the spread option, the coefficients' δ values are simply $\delta_1 = -1$ and $\delta_2 = 1$, while the generalised strike $k = K + R'_1(0) - R'_2(0)$, and $r_i = R'_i - R'_i(0)$ for the prime rates defined in (19).

■ **Cross-correlation parameterisation.** To simply correlation notation, we suppose $t = 1$ and denote BMs without arguments by $W \equiv W(1)$, such that their averages are equal to the correlations $\mathbb{E}[W^2] = \mathbb{E}[\mathrm{d}W^2]/\mathrm{d}t$.

Cross-correlations between two vector BMs $W^{(1)}$ and $W^{(2)}$:

$$C_{ij}^{(W)}\,\mathrm{d}t = \mathbb{E}[\mathrm{d}W_i^{(1)}(t)\,\mathrm{d}W_j^{(2)}(t)]$$

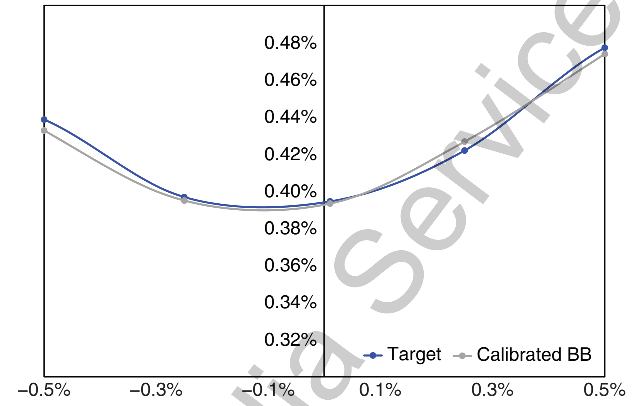
determine the calibration freedom for the option (20).

Obviously, if each rate's BMs are correlated, ie, $W_i^{(1)} = W^{(1)}$ and $W_i^{(2)} = W^{(2)}$, the spread will be driven by a single number – a correlation $\mathbb{E}[W^{(1)} W^{(2)}]$. Such a construction is equivalent to the Gaussian copula.

In the general case, these two Gaussian variables are the main drivers in option pricing: a correlation between them is the main calibration 'handle'. Thus, it is useful to express the vector BMs $W^{(1)}$ and $W^{(2)}$ in terms of internally uncorrelated $Z^{(1)}$ and $Z^{(2)}$ such that their first components are proportional to the main Gaussians $Z_1^{(1)} \sim G^{(1)}$ and $Z_1^{(2)} \sim G^{(2)}$. We denote the transformation matrix as $Q^{(h)}$:

$$W_i^{(h)} = \sum_{i'} Q_{ii'}^{(h)} Z_{i'}^{(h)}, \quad \text{where } \mathbb{E}[Z_i^{(h)} Z_{i'}^{(h)}] = \delta_{ii'} \text{ for } h = 1, 2$$

2 Black basket calibration to a 10Y–2Y spread option



Normal implied vol (vertical axis) as function of $K - F$ (horizontal axis), ie, a strike K position with respect to the forward F

See Antonov (2020) for more details. After this transformation, the cross-correlation freedom is concentrated inside $Z^{(1)}$ and $Z^{(2)}$:

$$C_{ij}^{(Z)}\,\mathrm{d}t = \mathbb{E}[\mathrm{d}Z_i^{(1)}(t)\,\mathrm{d}Z_j^{(2)}(t)]$$

The correlation between the first two elements, $Z_1^{(1)}$ and $Z_1^{(2)}$, is the main correlation between two rates: it essentially controls the ATM level of the mid-curve or spread option. The other cross-correlations determine the finer structure of the smile.

Clearly, the cross-correlation matrix $C^{(Z)}$ cannot have arbitrary elements (less than unity by absolute value) but should satisfy the extra constraints that guarantee the correlation matrix of the union of $Z^{(1)}$ and $Z^{(2)}$ is non-negatively definite. For our basic case of two exponents per rate (the general case is addressed in Antonov (2020) using the Givens rotation matrices), we can prove that the 2×2 cross-correlation matrix $C^{(Z)}$ can be fully parameterised by four angles θ_{ij} for $i, j = 1, 2$:

$$C^{(Z)} = \begin{bmatrix} \sin\theta_{1,1} & \cos\theta_{1,1}\sin\theta_{1,2} \\ \cos\theta_{1,1}\sin\theta_{2,1} & \begin{aligned} &\cos\theta_{2,1}\sin\theta_{2,2}\cos\theta_{1,2} \\ &\quad -\sin\theta_{2,1}\sin\theta_{1,1}\sin\theta_{1,2} \end{aligned} \end{bmatrix} \tag{22}$$

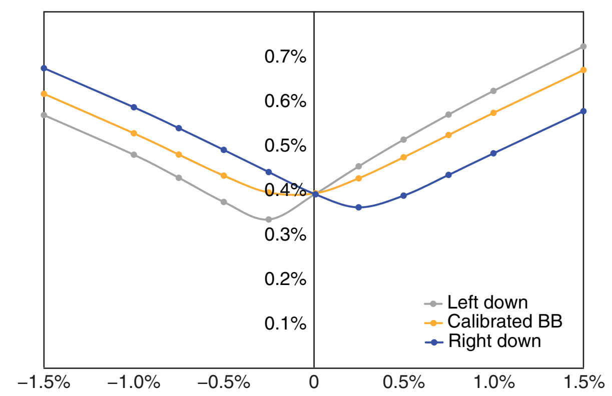
This guarantees a positive definition of the full correlation matrix $\rho^{(X)}$. The cross-correlation matrix between the initial BMs $W^{(1)}$ and $W^{(2)}$ can be calculated from $C^{(Z)}$ using the conversion formula $C^{(W)} = Q^{(1)} C^{(Z)} Q^{(2)\mathrm{T}}$.

■ **Calibration to spread options.** A calibration example for a mid-curve swaption is given in Antonov (2020). Here, we provide a more challenging experiment: a calibration to spread options where we have multiple quoted strikes and we need to have more degrees of freedom, ie, reacher 'copulas'. For this we decorrelate to maximise our internal BMs, fixing the internal correlations inside each standard rate to zero.

In our experiments we will concentrate on a 10Y–2Y spread option with 1Y maturity. Even for a 'compact' case of two-exponent rates, the calibration errors are very small (see figure 2, which shows the target (market) normal implied volatility and the resulting calibrated one).

In figure 3, we demonstrate a large smile flexibility with a 'pinned' ATM point. Specifically, we fix the ATM level to the market value and move out-of-the-money and in-the-money strikes in opposite directions. Then, we

3 Black basket calibration freedom for 10Y–2Y spread option



Normal implied vol (vertical axis) as function of $K - F$ (horizontal axis), ie, a strike K position with respect to the forward F

B. Calibrated cross-angles

	θ_{11}	θ_{12}	θ_{21}	θ_{22}
Left down	0.91	0.52	1.03	0
Calibrated BB	0.85	−0.25	0.16	0
Right down	0.8	−0.48	−1.25	0

analyse cross-correlation parameters calibrated to these targets (‘left down’, ‘calibrated BB’ and ‘right down’ in figure 3). In table B we observe that the main ‘handle’, $\theta_{1,1}$, is responsible for the ATM level (it varies slightly for the three experiments), while the others control the positions of the smile skew and wings. Qualitatively, large positive values of $\theta_{1,2}$ and $\theta_{2,1}$ lower the left wing, while large negative values lower the right wing.

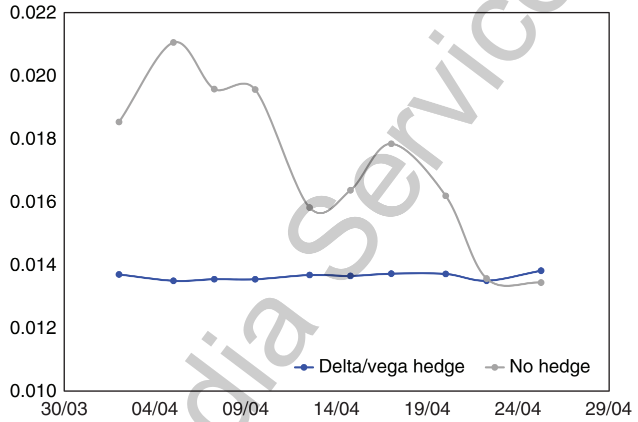
■ **Comments.** In this section we comment on our choices. First of all, in our numerical experiments we used a two-exponent Black basket setup (17). However, a larger number of exponents can provide a greater calibration freedom and a smile-independent wing control. Second, our choice of 1Y maturity is related to the most interesting form of smile for a relatively large data set. Indeed, for longer maturities, the smiles are pretty flat, and exercising the spread options’ calibration is not very challenging. Next, the speed matters in the calibration to multi-strike spread options. As mentioned in the appendix of Antonov (2020), an accurate procedure by Carmona & Durrleman (2006) involves at least a five-dimensional numerical optimiser, which can be slow. We propose an alternative procedure including only several iterations (see the appendix in Antonov (2020)).

Stability: parameters and hedging

A common practice in the financial industry is to recalibrate models when the market data get updated (say, daily). Of course, after such recalibrations the model parameters, being adapted to a different calibration target, change. An attractive property of a model is its recalibration stability, ie, the model parameters’ recalibration changes are small or moderate.

For numerical experiments we take a unit notional mid-curve swaption 1Y 5Y 5Y with a strike of 10 basis points (as of April 4, 2020) and move the observation date t through a one-month period from April 4 to May 4, 2020. The running time-to-maturity lies between 9M and 1Y – two standard expiries.

4 Movements of the mid-curve and hedged portfolio prices (vertical axis, in euros) as a function of the observation date (horizontal axis)



We use the Black basket models with two exponents per standard rate (17). For the running observation date t we build two Black basket models with maturities of 9M and 1Y. These are calibrated, respectively, to (1) underlying standard European swaptions 9M5Y/9M10Y and 1Y5Y/1Y10Y (three strikes each); and (2) ATM mid-curve swaptions 9M 5Y 5Y and 1Y 5Y 5Y. Then, we interpolate the model coefficients linearly in maturities to obtain the price of our mid-curve swaption (with a fixed maturity of April 4, 2021):

$$\mathcal{O}(t) = \mathcal{O}(\theta(t), \gamma(t), \sigma(t)) \qquad (23)$$

where $\theta(t)$ are cross-correlation angles (22), while $\gamma(t)$ and $\sigma(t)$ simply denote all the standard rates values and implied volatilities, respectively, used in the calibration.

The dependence (23) is in fact the generalised spread price (20) as a function of the cross-correlation angles θ and the standard rates’ parameters (21) that are expressed via the interest rates and their volatilities. The angle $\theta(t)$’s time dependence is a measure of how the spread Black basket model fits the market. Numerical experiments (see Antonov 2020) show the very mild time dependence of the main angle $\theta_{1;1}(t)$: its changes over one month do not exceed 0.025 radians. Given the derivative $\partial\mathcal{O}(t)/\partial\theta_{1,1}(t)$ of order around 0.002, the $\theta_{1,1}(t)$ noise results in a very tiny price change of 0.5bp.

As the second part of our experiments, we analyse the stability of a hedging portfolio. Namely, at the settlement date of April 4, 2020, we create a hedge of two underlying forward started swaps and European options (1Y5Y and 1Y10Y for three different strikes) such that the portfolio sensitivities to the above instruments are zero at inception. Then, we follow the portfolio price for one month – without re-hedging – and compare it with the ‘naked’ mid-curve swaption.

Figure 4 shows a dramatic reduction in the realised (normal) volatility, from 0.7% of the uncovered mid-curve swaption to 0.05% of the hedged one. This experiment shows that a frequent hedging of the mid-curve is not necessary: the model remains stable for one month.

The Black basket term structure model: generalising the LMM

Finally, we will sketch an extension of the Black basket spread option pricing to a full term structure modelling that permits pricing of arbitrary instruments and value-adjustment calculations to be performed. As an example, we provide a Black basket generalisation of the LMM (BB-LMM).

We introduce a set of forward Libors:

$$L_n(t) = \frac{P(t, T_n)/P(t, T_{n+1}) - 1}{\delta T_n}$$

defined on a schedule $\{T_n\}_{n=1}^{N}$ with the tenor $\delta T_n = T_{n+1} - T_n$. Then, for each Libor index n, consider a set of shifted lognormal processes in the T_{n+1}-forward measure:

$$Q_{n,b}(t) = \exp\biggl(\int_0^t \mathrm{d}W_{n,b}^{(T_{n+1})}(s)\,\sigma_{n,b}(s) - \frac{1}{2}\int_0^t \mathrm{d}s\,\sigma_{n,b}^2(s)\biggr) - 1 \tag{24}$$

for a set of time-dependent volatilities $\sigma_{n,b}(t)$. Indices b run over Black basket components and depend on the concrete Libor $b = 1, \dots, B_n$. We suppose general correlations between all the Brownians $\mathrm{d}W_{n,b}\,\mathrm{d}W_{n',b'} = \rho_{nn',bb'}\,\mathrm{d}t$. Of course, for better decorrelation freedom we can set zero correlations inside each Libor, $\mathrm{d}W_{n,b}\,\mathrm{d}W_{n,b'} = \delta_{bb'}\,\mathrm{d}t$. Now define the forward Libor evolutions as:

$$L_n(t) = L_n(0) + \sum_{b=1}^{B_n} \beta_{n,b} Q_{n,b}(t \wedge \tau_n) \tag{25}$$

where τ_n is the stopping time, when $L_n(t)$ touches a fairly low negative boundary l_n, and $\beta_{n,b}$ are the Black basket weights, which generalise the shifted LMM beta-shifts. We can eventually make them time-dependent but their 'flat' setup is already enough. Clearly, $L_n(t)$ is a martingale in the T_{n+1}-forward measure: the stopped martingale is a martingale.

Note that we have added the stopping property to a sum of Q values in order to limit the Libors from below, ie, guarantee that $L_n(t) \geqslant l_n$. The reason for this is to have a natural measure change process:

$$\frac{P(t, T_n)}{P(t, T_{n+1})} = 1 + L_n(t)\delta T_n$$

that is positive. For this it is sufficient to require the lower barrier to be $l_n > -1/\delta T_n$. This boundary (eg, for 3M Libor $l_n \sim -400\%$) is hardly attainable in a reasonable time for current interest rate volatilities.

To simulate the BB-LMM in a fixed model measure we apply the standard arguments. Given the Libor stochastic differential equation (SDE) in its forward measure:

$$\mathrm{d}L_n(t) = 1_{t<\tau_n} \sum_{b=1}^{B_n} \beta_{n,b}(1 + Q_{n,b}(t))\sigma_{n,b}(t)\,\mathrm{d}W_{n,b}^{(T_{n+1})}(t) \tag{26}$$

we can easily obtain the process $P(t, T_n)/P(t, T_{n+1})$ SDE to get the elementary measure change between two adjacent forward measures:

$$\mathrm{d}W_{m,a}^{(T_{n+1})}(t) = \mathrm{d}W_{m,a}^{(T_n)}(t) + \frac{1_{t<\tau_n}\sum_{b=1}^{B_n} \beta_{n,b}(1 + Q_{n,b}(t))\sigma_{n,b}(t)\rho_{nm,ba}^{(T_{n+1})}(t)}{1 + L_n(t)\delta T_n}\,\mathrm{d}t \tag{27}$$

One of the advantages of the BB-LMM is its highly analytical capabilities. For example, we can easily approximate an incremental swap rate $R(t)$ as a linear combination of the model Libor rates (in the R-martingale measure) $\mathrm{d}R(t) \simeq \sum_n w_n \mathrm{d}L_n(t)$, where the deterministic weights w_n are calculated using the yield curve information. We can further approximate it by removing the stopping time indicator $1_{t<\tau_n}$ due to its extreme rarity and finally obtain the rate in its pure Black basket form:

$$R(t) \simeq R(0) + \sum_n w_n \sum_{b=1}^{B_n} \beta_{n,b}\biggl(\exp\biggl(\int_0^t \mathrm{d}W_{n,b}(s)\sigma_{n,b}(s) - \frac{1}{2}\int_0^t \mathrm{d}s\,\sigma_{n,b}^2(s)\biggr) - 1\biggr) \tag{28}$$

We then apply the Black basket formula for the option pricing.

The BB-LMM generalises the shifted LMM (the latter is the BB-LMM special case for $B_n = 1$). The BB-LMM is simultaneously simple and flexible: it can be calibrated to multiple swaptions and CMS products. As a practical setup, we can consider a two-exponential case per Libor ($B_n = 2$). To address the adaptation of BB-LMM to the Libor reform, we can extend the Libor's evolution in the same way as for the standard LMM in Lyashenko & Mercurio (2019). Other details are left for future research.

Conclusions

In this paper we developed a new way to unify two fixed distributions such that their joint behaviour has a high degree of freedom. Moreover, the corresponding option on a linear combination of the two distributions can be calculated analytically. We applied this technique to mid-curve and spread options and demonstrated the clear practical advantages of this approach. We also sketched the Black basket extension of the LMM, which permits pricing of arbitrary instruments and value-adjustment calculations. ■

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