Libor replacement: a modelling framework for in-arrears term rates

Andrei Lyashenko and Fabio Mercurio define and model forward risk-free term rates – which appear in the payoff definition of derivatives and possibly cash instruments – based on the new interest rate benchmarks that are to replace Ibor globally. They show that the classic interest rate modelling framework can be naturally extended to describe the evolution of both the forward-looking and backward-looking term rates using the same stochastic process.

Ibor, which include Libor, Euribor, Tibor, Cdor and others, represent the cost of funds among global banks for short-term unsecured borrowing. They are the key reference rates in many financial products, with a total worldwide market exposure of more than US$370 trillion. Because the unsecured short-term lending market has not been sufficiently active lately, and because of reported widespread attempts to manipulate Libor during the 2007–9 crisis, the Financial Stability Board has recommended developing alternative nearly risk-free rates (RFRs) that are better suited to be the reference rates.

RFRs have now been selected in all major economies. For instance, the US has selected a new Treasuries repurchase agreement financing rate called the secured overnight financing rate (SOFR) and the United Kingdom has selected the reformed sterling overnight index average (Sonia). Since all the selected RFRs are overnight rates, in order for them to be used as replacements for Ibor they first need to be converted into term rates. Two main approaches are being considered by the International Swaps and Derivatives Association and by national regulators:
- a compounded setting-in-arrears rate, which is backward looking in nature and is known at the end of the corresponding application period; and
- a market-implied prediction of this compounded setting-in-arrears rate, which is forward looking in nature and is known at the beginning of the application period.

The first of these drove (and is driving) the definition of the new RFR futures and vanilla swaps, whereas the second seems to be favoured when it comes to defining fallbacks for cash products.

In this article, we show that by modelling the dynamics of term rates directly, we can simulate both forward-looking and backward-looking term rates using a single stochastic process for both. The joint modelling of these stochastic processes, for all the given application periods, leads to an extension of the classic single-curve Libor market model (LMM) that we call the generalised forward market model (FMM).

The FMM is a more complete model than the LMM as it preserves the dynamics of the forward-looking (Libor-like) rates while providing additional information about the rate dynamics between the term-rate fixing/payment times. It also has some nice properties the traditional LMM does not have, such as model-implied dynamics of forward rates under the classic money-market (‘continuous-spot’) measure and not only under the discrete spot measure as in the LMM case. Because of this, and because of its ability to handle financial quantities that depend on integrals of the short rates, the FMM should be implemented and used regardless of the discontinuation of Libor and the transition to new rate benchmarks.

Our FMM formulation is based on the concept of extended zero-coupon bonds, which we introduce: a concept that proves to be very convenient when dealing with backward-looking setting-in-arrears rates. Thanks to our extended definition, not only the bonds themselves but also the forwards and swap rates, along with the associated forward measures, can be extended and defined at all times, even those beyond their natural expiries.

Main assumptions, definitions and notation

We consider a continuous-time financial market with an instantaneous risk-free rate, whose time-t value is denoted by \( r(t) \). We assume rate \( r(t) \) is the collateral rate for collateralised over-the-counter transactions, as well as being the price alignment interest rate for cleared derivatives.\(^1\) This is consistent with the overall direction of the Ibor reform and the transition to the new rate benchmarks. If the same RFR is used for discounting and for deriving term rates, this implies a return to the classic single-curve modelling environment.

Rate \( r(t) \) has an associated money-market (or bank) account \( B(t) \) such that \( B(0) = 1 \) and:

\[
dB(t) = r(t)B(t)\,dt
\]

so:

\[
B(t) = e^{\int_0^t r(u)\,du}
\]

We assume the existence of a risk-neutral measure \( Q \), whose associated numeraire is \( B(t) \), and we denote by \( \mathbb{E} \) the expectation with respect to \( Q \) and by \( \mathcal{F}_t \) the ‘information’ available in the market at time \( t \): that is, the sigma-algebra generated by the model risk factors up to time \( t \). We then denote by \( P(t, T) \) the price at time \( t \) of the risk-free zero-coupon bond with maturity \( T \), that is:

\[
P(t, T) = \mathbb{E}\left[ e^{-\int_t^T r(u)\,du} \mid \mathcal{F}_t \right]
\]

which is defined for \( t \leq T \), it being the value of a contract that expires at time \( T \). However, the definition of \( P(t, T) \) can be extended to times \( t > T \)

\(^1\) A more general case where risk-free and collateral (or price alignment interest) rates are different has been considered by Mercurio (2018). The results in this article can easily be generalised to the case with different rates, but doing so comes at the expense of more complex notation and formulas.
as follows. Using (2) and the definition of $B(t)$, when $t > T$ we can write:

$$P(t, T) = \mathbb{E}[e^{\int_T^t r(u) du} \mid \mathcal{F}_T] = e^{\int_T^t r(u) du} \frac{B(t)}{B(T)}$$

(3)

since $\int_T^t r(u) du$ is $\mathcal{F}_T$-measurable. In particular, we have $P(t, 0) = B(t)$, meaning the money-market account can be viewed as a zero-coupon bond expiring immediately (that is, with 0 maturity).

The extended bond price $P(t, T)$ can be viewed as the time-$t$ value of the self-financing strategy that consists of buying the zero-coupon bond with maturity $T$ and reinvesting the proceeds of the bond’s unit notional at the risk-free rate $r(t)$ from time $T$ onwards.

- **Extended $T$-forward measure.** The extended zero-coupon bond price $P(t, T)$ continues to be a viable numeraire since it is the value of a self-financing strategy and is strictly positive. Therefore, we can define the (extended) $T$-forward measure $Q^T$ in the usual way as the equivalent martingale measure associated with the extended bond price $P(t, T)$. In contrast with the classic definition of forward measures, $Q^T$-dynamics can now be defined for any time $t$: that is, for times beyond maturity $T$ as well. The extended $T$-forward measure $Q^T$ is a hybrid measure that combines the classic $T$-forward measure up to the maturity time $T$ with the risk-neutral money-market measure $Q$ after $T$. Hybrid measures have been discussed by Glasserman & Zhao (2000) and Andersen & Piterbarg (2010), among others.

Since $P(t, 0) = B(t)$, the risk-neutral money-market measure is a particular case of the extended $T$-forward measure, where $T = 0$: $Q = Q^0$.

- **The market risk-free term rate.** Current derivatives contracts—such as futures, swaps and basis swaps—written on RFRs, such as SOFR or reformed Sonia, all reference the daily compounded setting-in-arrears rate based on the corresponding overnight benchmark. Recently, this rate was also chosen by Isda as the best risk-free term rate in the new Libor fallback definition. Therefore, modelling such term rates came to be of paramount importance for several reasons.

In what follows, we assume a time structure $0 = T_0, T_1, \ldots, T_M$, and we denote by $\tau_j$ the year fraction for the interval $[T_{j-1}, T_j)$. For each time $t$, we define $\eta(t) = \min\{j : T_j \geq t\}$, which is the index of the element of the time structure that is closest to time $t$ being equal to or greater than $t$. For brevity, we then use the shorthand notation $P_j(t)$ to denote the bond price $P(t, T_j)$.

For each $j = 1, \ldots, M$, we approximate the daily compounded setting-in-arrears rate for the interval $[T_{j-1}, T_j)$, which we denote by $R(T_{j-1}, T_j)$, as follows:

$$R(T_{j-1}, T_j) = \frac{1}{T_j} \left[ e^{\int_{T_{j-1}}^{T_j} r(u) du} - 1 \right]$$

(4)

In-arrears rates are backward looking in nature because one has to wait until the end of their accrual period to know their fixing value. Alternatively, one can define forward-looking rates, which are set at the beginning of their application period. For instance, a forward-looking rate at time $T_{j-1}$ with maturity $T_j$ can be defined similarly to an overnight indexed swap (OIS) rate. This rate, denoted by $F(T_{j-1}, T_j)$, is the fixed rate to be exchanged at time $T_j$ for the forward bank account $B(T_j) / B(T_{j-1})$ (minus 1 and divided by the year fraction) such that this swap has zero value at time $T_{j-1}$.

By no-arbitrage, we have:

$$F(T_{j-1}, T_j) = \mathbb{E}_T^P [R(T_{j-1}, T_j) \mid \mathcal{F}_{T_{j-1}}]$$

(5)

for each $j = 1, \ldots, M$.

- **Backward-looking in-arrears forward rates.** We define the backward-looking forward rate $R_j(t)$ at time $t$ to be the value of the fixed rate $K_R$ in the swaplet paying $t \cdot [R(T_{j-1}, T_j) - K_R]$ at time $T_j$ (see figure 1), such that the swaplet has zero value at time $t$.

By no-arbitrage, we have:

$$R_j(t) = \mathbb{E}_T^P [R(T_{j-1}, T_j) \mid \mathcal{F}_T]$$

(6)

From (5) and (6), we see that, for each $j = 1, \ldots, M$, the forward-looking spot rate $F(T_{j-1}, T_j)$ is equal to the backward-looking forward rate $R_j(t)$ at time $t = T_{j-1}$: $F(T_{j-1}, T_j) = R_j(T_{j-1})$.

A formula for the forward rate $R_j(t)$ can be derived by changing the measure to $Q$:² We obtain:

$$1 + t_j R_j(t) = \mathbb{E}_T^Q \left[ e^{\int_{T_{j-1}}^{T_j} r(u) du} \mid \mathcal{F}_T \right]$$

$$= \frac{1}{P_j(t)} \mathbb{E}_T^P \left[ e^{\int_{T_{j-1}}^{T_j} r(u) du} \mid \mathcal{F}_T \right] = \frac{P_{j-1}(t)}{P_j(t)}$$

(7)

so we can write:

$$R_j(t) = \frac{1}{t_j} \left[ \frac{P_{j-1}(t)}{P_j(t)} - 1 \right]$$

(8)

Notice that this is the classic, simply compounded, forward-rate formula, which holds for each time $t$, even those after $T_j$, thanks to our definition of the extended bond price.

The forward rate $R_j(t)$:

- is a martingale under the $T_j$-forward measure;
- is equal to the forward-looking spot rate at time $T_{j-1}$: $R_j(T_{j-1}) = F(T_{j-1}, T_j)$;
- is equal to the realised backward-looking rate at time $T_j$: $R_j(T_j) = R(T_{j-1}, T_j)$;
- stops evolving (that is, it is fixed) after time $T_j$: $R_j(t) = R(T_{j-1}, T_j)$, $t > T_j$.

When time $t$ is within the accrual period, that is, when $T_{j-1} < t < T_j$, the forward rate $R_j(t)$ ‘aggregates’ values of realised RFRs $r(s)$, $s \in [T_{j-1}, T_j]$.

² An equivalent derivation can be found in Mercurio (2018).
For each $j$, the forward rate $F_j(t)$ at time $t$ to be the value of the fixed rate $K_F$ in the swaplet that pays $\tau_j [F(T_{j-1}, T_j) - K_F]$ at time $T_j$, such that the swaplet has zero value at time $t$.

By no-arbitrage, we have, for $t < T_{j-1}$:

$$F_j(t) = \mathbb{E}^{T_j} [F(T_{j-1}, T_j) \mid \mathcal{F}_t] = \mathbb{E}^{T_j} [R(T_{j-1}, T_j) \mid \mathcal{F}_t] = R_j(t)$$

For $t > T_{j-1}$, the value is fixed and constant: $F_j(t) = F(T_{j-1}, T_j)$.

### Consolidating the two forward rates

For each $j = 1, \ldots, M$, the backward-looking forward rate $R_j(t)$ and the forward-looking forward rate $F_j(t)$ can be expressed by a single rate, which with some abuse of notation we will denote by $R_j(t)$. In fact, when $t \leq T_{j-1}$, the two rates are equal and are described by a single common value, $R_j(t)$. At time $t = T_{j-1}$, the forward-looking rate becomes fixed, $R_j(T_{j-1}) = F_j(T_{j-1}, T_j)$, and stops evolving. Instead, the backward-looking forward rate $R_j(t)$ continues its journey until it becomes fixed at time $T_j$. Finally, as already pointed out, $R_j(t) = R_j(t)$, $t > T_{j-1}$.

In the following, we will model the joint evolution of rates $R_j(t)$ for $j = 1, \ldots, M$ and introduce a natural extension of the classic single-curve LMM to the case where rates are set in arrears. The advantage of this approach is that we can obtain both forward-looking and backward-looking fixings using a single process.

### The dynamics of the forward rate

Because of its own definition (6), the forward rate $R_j(t)$ is a martingale under the corresponding $T_j$-forward measure, $j = 1, \ldots, M$. Its dynamics can be general for $t < T_{j-1}$, but its volatility must progressively decrease down to zero in its accrual period $[T_{j-1}, T_j]$, since it is proportional to the prompt bond volatility within that period. We then assume that:

$$dR_j(t) = \sigma_j(t) g_j(t) dW_j(t)$$

where, for each $j = 1, \ldots, M$, $\sigma_j(t)$ is an adapted process, $W_j(t)$ is a standard Brownian motion such that $dW_j(t) dW_j(t) = \delta_{j,j} dt$, and $g_j(t)$ is a (piecewise) differentiable (deterministic) function such that $g_j(t) = 1$ for $t \leq T_{j-1}$, $g_j(t)$ is monotonically decreasing in $[T_{j-1}, T_j]$ and $g_j(t) = 0$ for $t \geq T_j$. For instance, in the Ho–Lee model, the function $g_j$ is piecewise linear and is given by:

$$g_j(t) = \min \left\{ \frac{(T_j - t)^+}{T_j - T_{j-1}}, 1 \right\}$$

An empirical confirmation of the volatility decay of rates $R_j(t)$ is provided in figure 3, where we plot the daily absolute changes of the backward-looking forward rate whose application period started on June 20, 2018 and ended on September 18, 2018.

We stress again that, contrary to the classic LMM case, this dynamics is defined for any time $t$. Rate $R_j$ does not stop at time $T_{j-1}$ but continues to evolve stochastically until $T_j$, and remains constant thereafter. A plot of simulated paths of $R_j(t)$ under lognormal dynamics is shown in figure 4, where the effect of decaying volatility in the rate accrual period (the last quarter) is clearly visible.

### The generalised FMM

Equation (10) defines the dynamics of each forward rate $R_j(t)$ under the corresponding $T_j$-forward measure. A market model where all forward rates, for $j = 1, \ldots, M$, are modelled jointly can be defined by deriving the dynamics of each forward rate under a common probability measure (equivalently, numeraire). To this end, we apply the change-of-numeraire formula to derive the dynamics of $R_j(t)$ under a measure $Q^N$ associated with numeraire $N(t)$ (see, for example, Brigo & Mercurio 2006). Assuming continuous dynamics, the drift of $R_j$ under $Q^N$, as a function of time $t$, is given by:

$$\text{Drift}(R_j; Q^N(t)) = \frac{dR_j(t) d\ln(N(t) / P(t, T_j))}{dt}$$

We will consider three cases: (1) the money-market measure $Q$; (2) the discrete money-market (‘spot-Libor’) measure $Q^d$; and (3) a general $T_k$-forward measure $Q^{T_k}$.

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3 We refer to appendix A of Lyashenko & Mercurio (2019) for a formal justification of this assumption in a Heath-Jarrow-Morton one-factor framework.
Simulated paths of $R_j(t)$, where $T_{j-1} = 9M$ and $T_j = 1Y$, for $t \in [0, T_j]$, under lognormal dynamics with volatility equal to 30% and $R_j(0) = 2.5%$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{simulated_paths.png}
\caption{Simulated paths of $R_j(t)$, where $T_{j-1} = 9M$ and $T_j = 1Y$, for $t \in [0, T_j]$, under lognormal dynamics with volatility equal to 30% and $R_j(0) = 2.5%$.}
\end{figure}

\section*{Forward-rate dynamics under $Q^d$.}

We now apply (12) to the case where $N(t) = B(t)$ and $Q^N = Q$. The drift of $R_j$ under $Q$ is then given by:

$$\text{Drift}(R_j; Q(t)) = \frac{dR_j(t) \ln(B(t)/P(t; T_j))}{dt}$$

In a classic LMM, the risk-neutral dynamics of forward rates can be derived provided we also model the volatility of the prompt zero-coupon bonds $P(t; T_{k+1})$ (see, for example, Brigo & Mercurio 2006). Here, this extra assumption on the volatility of $P(t; T_{k+1})$ is no longer needed because it is implicit in the definition of the volatility of $R_j(t)$ in its corresponding accrual period.

Thanks to the definition of extended bond prices, we can write:

$$\ln \frac{B(t)}{P(t; T_j)} = \ln \frac{P(t; 0)}{P(t; T_j)} = \sum_{i=1}^{j} \ln \left[ 1 + 1 + \tau_i R_i(t) \right]$$

leading to:

$$\text{Drift}(R_j; Q(t)) = \sigma_j(t) g_j(t) \sum_{i=1}^{j} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)}$$

The Q-dynamics of $R_j$ then becomes:

$$dR_j(t) = \sigma_j(t) g_j(t) \sum_{i=1}^{j} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^Q(t)$$

where $W_j^Q$ is a $Q$-Brownian motion.

Because of the definition of $g_j(t)$, which is zero for $t > T_i$, that is, for $\eta(t) > i$, the drift term in (14) can also be expressed in terms of the index function $\eta(t)$. We have:

$$dR_j(t) = \sigma_j(t) g_j(t) \sum_{i=\eta(t)}^{j} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^Q(t)$$

\section*{Forward-rate dynamics under $Q^d$.}

We now apply (12) to the case where:

$$N(t) = B_d(t) = \frac{P(t; T_{k+1})}{1 + \sum_{i=1}^{k} \tau_i R_i(t)}$$

and $Q^N = Q^d$. We have:

$$\text{Drift}(R_j; Q^d(t)) = dR_j(t) \ln(P(t; T_{k+1}))/P(t; T_j))$$

A derivation similar to that in the previous section then leads to:

$$dR_j(t) = \sigma_j(t) g_j(t) \sum_{i=\eta(t)+1}^{j} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^d(t)$$

where $W_j^d$ is a standard Brownian motion under $Q^d$, and where the drift is zero when $j \leq \eta(t)$.

Comparing (15) with (16), we see that the difference between the rate dynamics under $Q$ and $Q^d$ is given by the following drift adjustment:

$$\text{Drift}(R_j; Q^d(t)) = \text{Drift}(R_j; Q(t)) + \sum_{i=1}^{j} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^d(t)$$

In the classic LMM, when approximating $Q$ with $Q^d$, one cannot quantify the magnitude of the approximation. This issue can be addressed by our generalised FMM. The impact of moving from $Q$ to $Q^d$ is represented by (17) and can be measured accordingly.

Note also that since $g_i(T_i) = 0$, the $Q$-drift term in (15) does not experience a jump when time $t$ moves right past $T_i$ and index $\eta(t)$ jumps from $i$ to $i+1$. On the other hand, the $Q^d$-drift in (16) does jump, since it loses the $(i+1)$-term.

\section*{Forward-rate dynamics under $Q^{Tk}$.}

Finally, we apply (12) to the case where $N(t) = P(t; T_k)$ and $Q^N = Q^{Tk}$. The drift of $R_j$ under $Q^{Tk}$ is thus given by:

$$\text{Drift}(R_j; Q^{Tk}(t)) = \frac{dR_j(t) \ln(P(t; T_k)/P(t; T_j))}{dt}$$

Repeating the same procedure as before, we then get:

$$dR_j(t) = \sigma_j(t) g_j(t) \sum_{i=\eta(t)+1}^{k} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^k(t)$$

when $j > k$, and:

$$dR_j(t) = -\sigma_j(t) g_j(t) \sum_{i=\eta(t)+1}^{k} \frac{\tau_i \sigma_i(t) g_i(t)}{1 + \tau_i R_i(t)} dt + \sigma_j(t) g_j(t) dW_j^k(t)$$

when $j < k$, where $W_j^k$ is a standard Brownian motion under $Q^{Tk}$.

Remark 1 Since $Q^{Tk}$ is a hybrid measure that consists of the classic $T_k$-forward measure up to time $T_k$ and of the risk-neutral measure $Q$ after $T_k$, we should have:

$$\text{Drift}(R_j; Q^{Tk}(t)) = \text{Drift}(R_j; Q(t))$$

for $t \geq T_k$.

Using the drift formulas from the previous sections, we can prove this is indeed the case. In particular, when $k = 0$, we confirm that:

$$\text{Drift}(R_j; Q^0(t)) = \text{Drift}(R_j; Q(t))$$
Differences between FMM and LMM

As we have seen, the FMM is an extension of the classic single-curve LMM in that it models the joint dynamics not only of adjacent (simply compounded) forward-looking forward rates $F_j(t)$, as in the LMM, but also of backward-looking (setting-in-arrears) forward rates $R_j(t)$, since $F_j(t) = R_j(t)$ for all times $t$ before the expiry time $T_{j-1}$ of $F_j(t)$. On top of this, the FMM has additional properties, which we summarise as follows.

- **Model completeness.** The main property that distinguishes the generalised forward rates $R_j(t)$ from Libor is their completeness in terms of spanning the periods defined by the time grid $T_0, \ldots, T_M$. Indeed, for any index $j = 1, \ldots, M$ and for any time $t$, we can express the price of a zero-coupon bond with maturity $T_j$ in terms of the bank account $B(t)$ and forward rates $R_j(t)$ as follows:

  \[ P(t, T_j) = B(t) \sum_{i=1}^{j} \frac{1}{1 + R_j(t)} \]

  with the equality holding for all $t$, including $t > T_j$. This implies that the volatility of all bonds $P(t, T_j)$ is known and is a function of rates $R_j(t)$ as well as their instantaneous covariance structure.

  Analogous representations are not available in the LMM, that is, when using forward-looking forward rates $F_j(t)$. This is exactly the reason why we have a closed-form drift term representation under the $Q$-measure for FMM, but not for LMM. In this respect, FMM is a complete model while LMM is not.

  The availability of $Q$-dynamics of term forward rates presents a number of advantages when it comes to the valuation of general derivatives, which we describe below.

- **Better pricing of futures contracts.** As shown by Hunt & Kennedy (2000), the time-$t$ futures price of a contract that pays out $H_T$ at time $T > t$ is given by:

  \[ f(t) = E[H_T \mid F_t] \]

  In a classic LMM, since no $Q$-dynamics are directly available, one typically approximates $Q$ with $Q^d$ to explicitly calculate the futures price, $f(t) \approx E^d[H_T \mid F_t]$, where $E^d$ denotes expectation under $Q^d$. This approximation is no longer needed for the FMM, since we know the forward-rate dynamics under $Q$, so the former formula can be used.

- **Easier extension to a cross-currency interest rate model.** Assume we have a two-currency economy where domestic and foreign rates are driven by corresponding FMMs, and denote by $X(t)$ the spot exchange rate at time $t$, meaning one unit of foreign currency can be purchased with $X(t)$ units of domestic currency. By modelling the dynamics of $X(t)$, we can easily derive the dynamics of the foreign FMM under the domestic measure $Q$ as well as the dynamics of the domestic FMM under the foreign money-market risk-neutral measure $Q^d$.

  In fact, assuming continuous dynamics, the drift of the foreign rate $R^d_j(t)$ under the domestic measure $Q$ is given by:

  \[ \text{Drift}(R^d_j; Q)(t) = \text{Drift}(R^d_j; Q^d)(t) - \frac{d_R^d_j(t) \, d \ln(X(t))}{dt} \]

  An analogous formula applies for the drift of $R_j(t)$ under $Q^d$. Again, similar formulas are not available in the classic LMM.

- **More natural hybrid modelling.** In the case of, for instance, a hybrid equity-interest rate model, the equity risk-neutral drift rate (assuming no dividends) is equal to $r(t)$, which is not known in the classic LMM, and nor are its integrals. When using an FMM instead, we can express the integral of the drift rate (at points $T_j$) in terms of rates $R_j(t)$, which are by definition known in the model. To illustrate this, assume the $Q$-dynamics of a given stock $Z$ is:

  \[ dZ(t) = r(t)Z(t) \, dt + \sigma Z(t) \, dW(t) \]

  Then, for each pair of indexes $j < k$, we can write:

  \[ Z(T_k) = Z(T_j) \prod_{i=j+1}^{k} \left[ 1 + \frac{1}{t_i} R_i(t) \right] e^{-\frac{1}{2} \sigma^2 \left( T_k - T_j \right) + \sigma \left( W(T_k) - W(T_j) \right)} \]

  Therefore, $Z(t)$ can be simulated by simulating the joint evolution of rates $R_j(t)$ and Brownian motion $W(t)$, assuming a correlation structure among them.

- **Valuation of RFR vanilla derivatives.** SOFR and Sonia futures are currently traded on CME and Ice. LCH and CME started clearing SOFR and Sonia fixed-floating swaps, which are very similar to OIS contracts. At the time of writing, there is still no trading activity on RFR caps or swaptions.

  **The valuation of RFR futures.** Consider a futures contract where the underlying rates are the daily compounded RFRs, which we approximate by $R(T_{j-1}, T_j)$. The corresponding futures convexity adjustment at time $t$ is given by:

  \[ C_f(t) = E[R(T_j) \mid F_t] - R(t) \]

  Using (14), we can then write:

  \[ C_f(t) = \int_t^{T_j} E[\text{Drift}(R_j; Q)(s) \mid F_t] \, ds \]  

  The $Q$-drift of $R_j(t)$ generally depends on $R_j(t)$ itself as well as on other forward rates $R_k(t)$, which makes it impossible, in general, to calculate the expectation in (20) in exact closed form. However, an explicit approximation for $C_f(t)$ can be derived for any rate dynamics by using a standard drift-freezing technique; see Lyasenko & Mercurio (2019) for further details.

  **The valuation of an RFR fixed-floating swap.** The RFR fixed-floating swaps are identical in their structure to the current OIS contracts, where the floating leg pays daily compounded setting-in-arrears RFRs. Therefore, the swap value to the party receiving floating-rate payments on dates $T_{a+1}, \ldots, T_b$ and paying the fixed rate $K$ on dates $T'_c, \ldots, T'_d$, with $T'_c = T_a$ and $T'_{d+} = T_b$, is equal to \[ S(t) - K \cdot A(t), \] where $S(t)$ and $A(t)$ are the swap rate and the annuity, respectively, defined using the standard (single-curve) formulas:

  \[ S(t) = \frac{P(t, T_a) - P(t, T_b)}{A(t)} \]

  \[ A(t) = \sum_{j=c+1}^d t_j' P(t, T_j') \]

  where $t_j'$ denotes the year fraction for the fixed-leg interval $[T'_{j-1}, T'_j]$.

  **Remark 2.** Since the (extended) zero-coupon bond prices $P(t, T_j)$ and forward rates $R_j(t)$ are defined for all values of time $t$, the swap price and rate formulas above are also defined for all values of $t$. For $t > T_a$, the swap value represents the time-$t$ value of a self-financing investment strategy where
all cashflows are reinvested (financed in the case of negative cashflows) at the risk-free rate to roll them forward to the present time.

- **The valuation of an RFR cap.** For each given $j$ and associated application period $[T_{j-1}, T_j)$, we can define two distinct caplets with strike $K$ that pay off at time $T_j$:
  - a forward-looking caplet with payoff $[R_j(T_{j-1}) - K]^+$; and
  - a backward-looking caplet with payoff $[R_j(T_{j-1}) - K]^+$. The main difference between these two payoffs is the former is known at the beginning of the application period, $T_{j-1}$, whereas the latter is known at the end, $T_j$. At the time of writing, it is still unclear which of the two payoffs will prevail in the market. However, there are valid reasons to presume both will be made available to customers.

The valuation of the two caplets relies on the modelling of the forward rate $R_j(t)$ in the $T_j$-forward measure. However, by the tower property of conditional expectations and the Jensen inequality, we have, for $t \leq T_{j-1}$:

$$E^{T_j}[\{R_j(T_j) - K\}^+ | \mathcal{F}_t] \geq E^{T_j}\{E^{T_j}[\{R_j(T_j) - K\}^+ | \mathcal{F}_{T_{j-1}}] - K\}^+ | \mathcal{F}_t\}
$$

$$= E^{T_j}\{E^{T_j}[\{R_j(T_{j-1}) - K\}^+ | \mathcal{F}_{T_{j-1}}]\}
$$

where we applied the martingale property of $R_j(t)$, that is:

$$E^{T_j}[R_j(T_j) | \mathcal{F}_{T_{j-1}}] = R_j(T_{j-1})$$

This implies the backward-looking caplet is always more expensive than the forward-looking one.$^4$

An example based on lognormal dynamics is considered in Lyashenko & Mercurio (2019). A formula based on a one-factor Heath-Jarrow-Morton model is provided by Henrard (2019).

- **The valuation of an RFR swaption.** An RFR swaption, either payer or receiver, can be defined as the option to enter a swap RFR swap on the swap’s maturity date. Like a Libor-based swap rate, an RFR swap rate is also a martingale under the forward swap measure associated with its annuity. We can then assume specific dynamics for the swap rate under this measure, and price swaptions accordingly. For instance, assuming a (driftless) geometric Brownian motion leads to the Black formula for swaptions.

$^4$ This is also intuitive: $R_j(T_j)$ and $R_j(T_{j-1})$ have the same mean but the former has a larger variance.

The valuation of swaptions in the market model where each forward rate evolves according to (10) is equivalent to the valuation of Libor-based swaptions in the single-curve LMM. We can then refer to the existing single-curve LMM literature for the closed-form approximations that can be used depending on the chosen dynamics.

**Conclusions**

In this article, we showed that setting-in-arrears backward-looking rates can be modelled jointly with the traditional forward-looking rates using the classic single-curve modelling framework by naturally extending the definition of a zero-coupon bond to times beyond its maturity. This led to our generalised FMM, an extension of the classic LMM, which has several important advantages over the LMM itself, such as the accessibility of the forward rate evolution under the (continuous) money-market measure and a higher resolution of the rate dynamics within each accrual period. Moreover, the implementation cost of switching from the LMM to the FMM should be a fraction of the total cost of building the FMM from scratch. We also expect the FMM performance to be comparable to that of an existing LMM. In fact, when simulating the FMM, we do not anticipate the need for a finer time grid, because the risk-neutral dynamics of each forward rate in its accrual period is given by a closed stochastic differential equation, which can allow for efficient variance-reduction techniques. Our expectation is also that the same numerical approaches used for the LMM could be applied to the FMM.

Our framework can also be enhanced by adding risky Libor-like rates, as in Mercurio (2010). A multi-curve market model can be built by modelling the joint covariance structure of RFR term rates and Libors (or Libor proxies).

Andrei Lyashenko is the head of market risk and pricing models at Quantitative Risk Management (QRM) in Chicago and an adjunct professor at the Illinois Institute of Technology. Fabio Mercurio is global head of quantitative analytics at Bloomberg LP, New York and an adjunct professor at NYU. We thank Mark Henrard, Paul Glasserman and Peter Carr for their comments and suggestions. Andrei Lyashenko thanks his colleague Yutian Nie for initial insight. The views and opinions expressed in this article are our own and do not represent the opinions of any firm or institution. Bloomberg LP and Bloomberg Professional Services are trademarks and service marks of Bloomberg Finance LP, a Delaware limited partnership, or its subsidiaries. All rights reserved.

Email: andreii.lyashenko@qrm.com, fmercurio@bloomberg.net.

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