# Roughening Heston

Rough volatility models are known to fit the volatility surface with very few parameters. The classical Heston model, however, is highly tractable, allowing for fast calibration. Omar El Euch, Jim Gatheral and Mathieu Rosenbaum present here the rough Heston model, which combines these two worlds. In particular, the authors find they can accurately approximate rough Heston model values by scaling the volatility-of-volatility parameter of the classical Heston model



Rough volatility models have quickly grown in popularity because of their ability to consistently model both historical and implied volatilities with very few parameters. However, even with the introduction of the efficient hybrid Brownian semistationary (BSS) scheme of Bennedsen *et al* (2017), practical implementation has proved to be difficult. The recent advances of El Euch *et al* (2018) and El Euch & Rosenbaum (2018, 2019) have demonstrated how a natural rough generalisation of the Heston model emerges as the macroscopic limit of a simple high-frequency trading model. This reflects the persistence of order flow, the high degree of endogeneity of the market and the liquidity asymmetry between the bid and ask sides of the limit order book. In addition, the authors derive the characteristic function of the log price as well as hedging strategies in their model.

In this article, we present the rough Heston model and explain how to use it in practice. As in the rough Bergomi model of Bayer *et al* (2016), the forward variance curve $\xi_t(u) = \mathbb{E}[V_u \mid \mathcal{F}_t]$, where $V_u$ is the spot variance at time $u$, is a state variable such that the model can be made to match at-the-money volatilities exactly. We are left with only three parameters to calibrate (which are defined in the next section): the Hurst exponent $H$, the volatility of volatility $\nu$ and the correlation $\rho$ between spot moves and volatility moves.[1]

Taking the parameter $H$ as an example, (historical) $H$ may be estimated from scaling properties of historical volatility estimated from the time series of high-frequency asset log returns. Meanwhile, (implied) $H$ may be estimated from implied volatilities at a fixed time by looking at the scaling of the short-dated implied volatility skew with time to expiration. The two values of $H$ thus obtained are consistently small: typically of the order of 0.1. We can analyse the two other parameters, $\rho$ and $\nu$, similarly, and we find both are consistent between the time series of log returns and the implied volatility surface on a given date. Moreover, as is shown in Bayer *et al* (2016), the forward variance curve on a given date is consistent with the historical time series. Finally, the consistency of rough volatility models with the dynamics of implied volatilities is formally proved in Fukasawa (2017).

## The rough Heston model

The rough Heston model of El Euch & Rosenbaum (2019) for a one-dimensional asset price $S$ takes the form:

$$\frac{\mathrm{d}S_t}{S_t} = \sqrt{V_t}\Big\{\rho\,\mathrm{d}B_t + \sqrt{1-\rho^2}\,\mathrm{d}B_t^{\perp}\Big\}$$

with:

$$V_u = V_t + \frac{\lambda}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\theta^t(s)-V_s}{(u-s)^{(1/2)-H}}\,\mathrm{d}s + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-s)^{(1/2)-H}}\,\mathrm{d}B_s, \quad u \geqslant t \qquad (1)$$

where $H \in (0, \frac{1}{2})$ is the Hurst exponent, $\nu$ is the volatility of volatility, $\rho \in [-1, 1]$ is the correlation between spot and volatility moves, $\lambda \geqslant 0$ and $\Gamma$ denotes the gamma function. The mean reversion level parameter $\theta^t(\cdot)$ is an $\mathcal{F}_t$-measurable function that makes the model time consistent, as explained in El Euch & Rosenbaum (2018). It is straightforward to verify (1) gives back the classical Heston model with time-dependent mean reversion level in the limit $H \to \frac{1}{2}$. From El Euch & Rosenbaum (2019), volatility sample paths have Hölder regularity $H - \varepsilon$ for any $\varepsilon > 0$, hence the name 'rough Heston' being given to this model. It is also shown in El Euch & Rosenbaum (2018) that $\lambda\theta^t(\cdot)$ can be directly inferred from the forward variance curve $(\xi_t(u))_{u\geqslant t}$ observed at time $t$. By doing so, the model may be rewritten in the asymptotic setting $\lambda \to 0$ in forward variance form as:

$$\frac{\mathrm{d}S_t}{S_t} = \sqrt{V_t}\Big\{\rho\,\mathrm{d}B_t + \sqrt{1-\rho^2}\,\mathrm{d}B_t^{\perp}\Big\}$$

with:

$$V_u = \xi_t(u) + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_t^u \frac{\sqrt{V_s}}{(u-s)^{(1/2)-H}}\,\mathrm{d}B_s, \quad u \geqslant t \qquad (2)$$

Remark 1 Recall that the forward variance curve may in principle be obtained from the variance swap curve by differentiation. More practically, assuming continuous sample paths, it is well known that the fair value of variance swaps can be obtained from an infinite log strip of out-of-the-money options (see, for example, Gatheral 2006).

## Pricing and hedging

Just as in the classical case, we can compute in quasi-closed form the characteristic function of the log price of the stock in the rough Heston model. This makes the model highly tractable and easy to calibrate.

We define the fractional integral of order $r \in (0, 1]$ of a function $f$ as:

$$I^r f(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)\,\mathrm{d}s$$

whenever the integral exists, and its fractional derivative of order $r \in [0, 1)$ as:

$$D^r f(t) = \frac{1}{\Gamma(1-r)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-r} f(s)\,\mathrm{d}s$$

[1] *From the rough volatility perspective, smiles flatten consistently with the scaling properties of historical volatility and, as discussed in Gatheral* et al *(2018), there is no need for an extra mean reversion parameter.*

whenever it exists. From El Euch & Rosenbaum (2018), the rough Heston model is Markovian in $X_t = \log(S_t)$ and the forward variance curve $(\xi_t(u))_{u \geqslant 0}$, in the same spirit as the Bergomi model (Bergomi 2005). In forward variance form, the characteristic function of the terminal log spot $X_T$ conditional on the time-$t$ initial state $(X_t, \xi_t)$ is given by:

$$\begin{aligned}\phi_t(T,a) &= \mathbb{E}_{X_t,\xi_t}[\exp\{\mathrm{i}aX_T\}] \\ &= \exp\left\{\mathrm{i}aX_t + \int_t^T D^\alpha h(a, T-u)\xi_t(u)\,\mathrm{d}u\right\}\end{aligned} \tag{3}$$

where $\alpha = H + \frac{1}{2}$ and $h(a,\cdot)$ is the unique continuous solution of the fractional Riccati equation:

$$\left.\begin{aligned} D^\alpha h(a,t) = -\tfrac{1}{2}a(a+\mathrm{i}) + \mathrm{i}\rho\nu a h(a,t) + \tfrac{1}{2}\nu^2 h^2(a,t) \\ I^{1-\alpha}h(a,0) = 0 \end{aligned}\right\} \tag{4}$$

Equation (4) is a rough version of the Riccati equation that arises in the classical Heston model (with zero mean reversion). Indeed, the only difference is that the time derivative is replaced by a fractional derivative. In contrast to the classical Heston case, there is no explicit solution to (4). This equation can be solved efficiently using numerical methods for fractional ordinary differential equations. We present one such method, the Adams scheme, in Appendix A; see Abi Jaber & El Euch (2018), Callegaro *et al* (2018) and Gatheral & Radoičić (2019) for newly developed alternative numerical methods. Moreover, as we explain later, the true solution may also be accurately approximated in closed form by a scaled version of the classical Heston solution. European option prices may then be obtained from the characteristic function using standard Fourier techniques (see, for example, Gatheral 2006).

With the characteristic function now in the Markovian form (3), hedging European options becomes obvious. Let $C_t(T) = \mathbb{E}[f(X_T)|\mathcal{F}_t]$. A European option with payoff $f(X_T)$ can then be perfectly replicated. As of time $t$, the hedge portfolio has $\partial_{S_t}C_t(T)$ of stock and $\partial_{\xi_t}C_t(T)$ of the forward variance curve $\xi_t(s)$ for each $s \in (t,T]$, where $\partial_{\xi_t}$ represents the Fréchet derivative: roughly speaking, the portfolio corresponding to bumping each of the forward variances $\xi_t(s)$ (see El Euch & Rosenbaum 2018). From the above expressions, it is clear that perfect replication is purely theoretical. In practice, as with interest rates, one holds a finite number of variance contracts. Note, however, that the rough Heston model has only one volatility factor. One can therefore hedge with only one European option, as in the classical Heston case, provided the value of the option component of the hedge portfolio coincides with the theoretical value of the forward variance component.

## Calibration of the rough Heston model

In this section, we present SPX volatility surface calibration results for two dates: August 14, 2013, to compare with the rough Bergomi calibration given for that day in Bayer *et al* (2016); and May 19, 2017, to show the model continues to fit the market very well.

Note the only parameters we calibrate are the Hurst parameter $H$, the volatility of volatility $\nu$ and the correlation $\rho$. The forward variance curve, being a state variable in the model, is fixed to match at-the-money volatilities.

■ **SPX calibration on August 14, 2013.** From Bloomberg, on August 14, 2013, there were 19 expirations from 1 day to over 2.5 years, for a total of 1,809 options quoted. After eliminating options for which the bid price (of either the put or the call) was zero, we are left with 1,290 strike-expiration pairs.

For each such strike-expiration pair, we compute bid and ask implied volatilities:

$$\sigma_i^{\pm} := \sigma_{\mathrm{BS}}^{\pm}(k_i, T_i)$$

where $k_i$ denotes the log strike. Given model parameters $\{H, \nu, \rho\}$, we can obtain model implied volatilities $\sigma_i^{\mathrm{M}}(H,\nu,\rho)$. We calibrate the parameters by minimising:[2]

$$\sum_i [\sigma_i^M(H,\nu,\rho) - \sigma_i^+]^2 + [\sigma_i^M(H,\nu,\rho) - \sigma_i^-]^2$$

subject to the constraints:

$$H \in (0, \tfrac{1}{2}], \quad \nu \geqslant 0, \quad \rho \in [-1, 1]$$

We obtain the following optimal parameters:

$$H = 0.1216, \quad \nu = 0.2910, \quad \rho = -0.6714$$

Figure 1 shows a remarkable fit to the SPX volatility surface, which is as good as with the rough Bergomi model in Bayer *et al* (2016).

■ **SPX calibration on May 19, 2017.** From Bloomberg, on May 19, 2017, there were 27 expirations from 1 day to over 2.5 years, for a total of 2,743 options quoted.

By applying the same calibration procedure, we obtain the following optimal parameters:

$$H = 0.0474, \quad \nu = 0.4061, \quad \rho = -0.6710$$

On this particular day, the calibrated value of $H$ is closer to zero and so corresponds to very rough volatility.

■ **Consistency with historical data.** In addition, we have that the calibrated Hurst parameter on August 14, 2013 (see the earlier section titled 'SPX calibration on August 14, 2013') is consistent with the one computed from historical data. Indeed, in Gatheral *et al* (2018), the authors show the behaviour of historical log volatility of the SPX index is close to that of a fractional Brownian motion with small Hurst parameter of order 0.1. As explained in detail in Gatheral *et al* (2018), if we estimate the moment of order $q$ of a log volatility increment over a time interval of length $\Delta$ by:

$$m(\Delta, q) = \frac{1}{N}\sum_{k=1}^{N} |\log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta})|^q$$

where the $(\sigma_{k\Delta})_{0 \leqslant k \leqslant N}$ are historical measurements of volatility, we obtain a strong linear relationship between $\log(m(\Delta, q))$ and $\log(\Delta)$. In summary, we find:

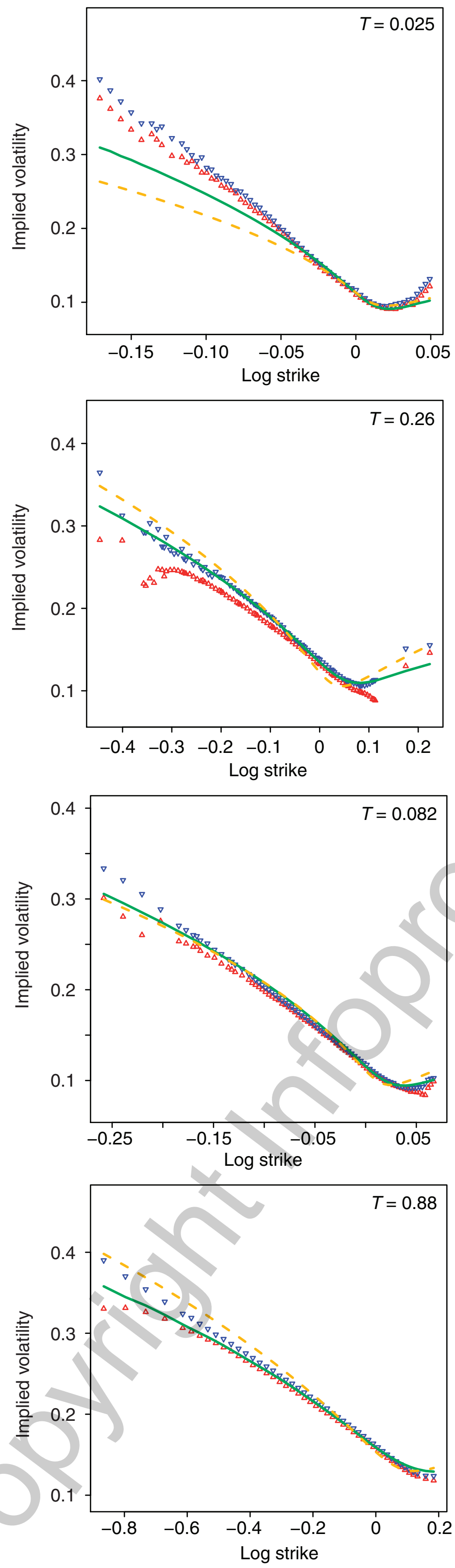
$$\mathbb{E}[|\sigma_\Delta - \sigma_0|^q] \approx K_q \Delta^{qH}, \quad H \approx 0.14$$

which is in theory what we have if the log volatility is a fractional Brownian motion with Hurst parameter $H = 0.14$.

Four representative SPX smiles as of August 14, 2013 and May 19, 2017 are plotted in figures 1 and 2 respectively. We note that on both of these days,

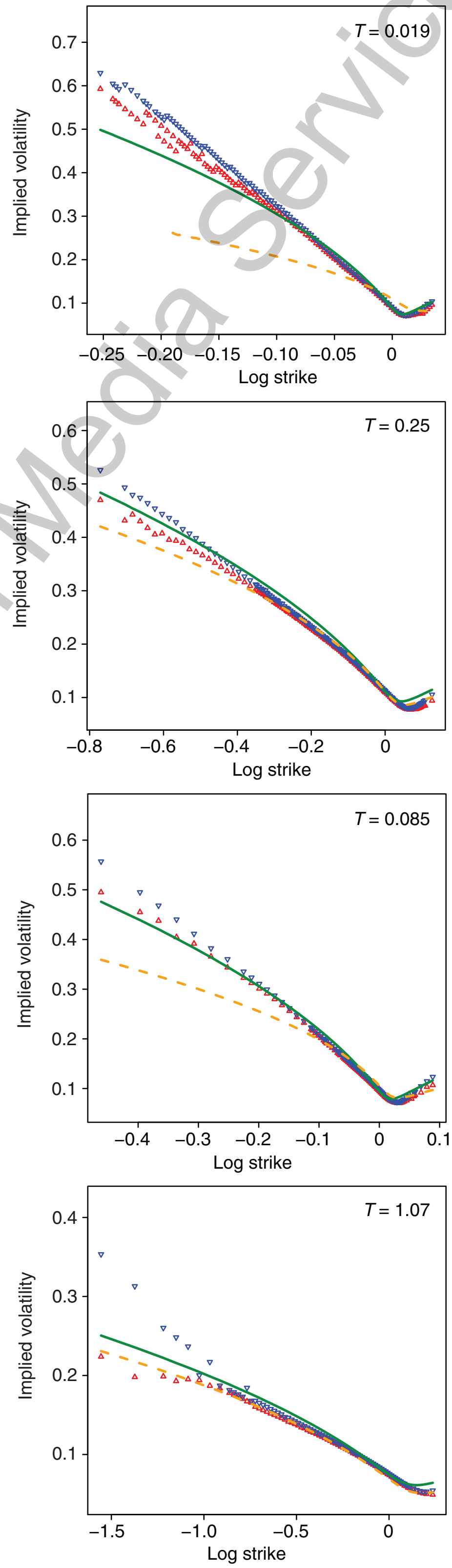
[2] *Alternatively, the parameters of the rough Heston model may be efficiently calibrated to the term structure of leverage swaps using a closed-form formula (see Alòs* et al *2017).*

1 Representative SPX volatility smiles as of August 14, 2013, with time to expiration in years




Red and blue points represent bid and ask SPX implied volatilities; green smiles are from the rough Heston model calibrated using the Adams scheme; dashed orange lines are from the classical Heston model fitted to these four smiles

2 Representative SPX volatility smiles as of May 19, 2017, with time to expiration in years




Red and blue points represent bid and ask SPX implied volatilities; green smiles are from the rough Heston model calibrated using the Adams scheme; dashed orange lines are from the classical Heston model fitted to these four smiles the calibrated values of $H$ are small, consistent with the small $H$ estimated from historical data. The rough Heston smiles are significantly closer to the market than the classical Heston smiles with optimised Heston parameters. Indeed, rough volatility models, though very parsimonious, are known to be consistent with historical data (see, for example, Gatheral *et al* 2018) and also, as seen in these figures, with implied volatility data.

Of course, the rough Heston model, with only three parameters, cannot possibly generate implied volatilities within the bid-ask spread for each strike and expiration. One way to keep the desirable features of a rough volatility model and yet ensure a perfect fit to the volatility surface would be to construct a stochastic local volatility model with the rough Heston model as the stochastic volatility backbone.

## A poor man's rough Heston model

In this section, we present fast and almost instantaneous approximation methods to compute the implied volatility for a given expiry $T$ and parameters $(H, \rho, \nu)$. The realised variance of the rough Heston model is given from (2) by:

$$\int_0^T V_u \, \mathrm{d}u = \int_0^T \xi_0(u)\,\mathrm{d}u + \frac{\nu}{\Gamma(H+\frac{3}{2})} \int_0^T (T-u)^{(1/2)+H} \sqrt{V_u}\,\mathrm{d}B_u \tag{5}$$

The variance of (5) is given by:

$$\nu^2 \int_0^T \frac{(T-s)^{2H+1}}{\Gamma(H+\frac{3}{2})^2} \xi_0(s)\,\mathrm{d}s \tag{6}$$

This suggests we approximate the rough Heston smile with the smile generated by a classical Heston-like model (2) with $H = \frac{1}{2}$ and with a scaled volatility-of-volatility parameter $\tilde{\nu}(T)$ matching the variance (6), that is:

$$\nu^2 \int_0^T \frac{(T-s)^{2H+1}}{\Gamma(H+\frac{3}{2})^2} \xi_0(s)\,\mathrm{d}s = \tilde{\nu}(T)^2 \int_0^T (T-s)^2 \xi_0(s)\,\mathrm{d}s$$

Thus:

$$\tilde{\nu}(T) = \frac{\nu}{\Gamma(H+\frac{3}{2})} \sqrt{\frac{\int_0^T (T-s)^{2H+1}\xi_0(s)\,\mathrm{d}s}{\int_0^T (T-s)^2 \xi_0(s)\,\mathrm{d}s}} \tag{7}$$

For each expiry $T$, the characteristic function formula (3) is then approximated by the classical one:

$$\exp\left\{\mathrm{i}aX_0 + \int_0^T \partial_u h^{(T)}(a, T-u)\xi_0(u)\,\mathrm{d}u\right\} \tag{8}$$

where $h^{(T)}(a,\cdot)$ is now a solution of the classical Riccati equation:

$$\begin{aligned}\partial_u h^{(T)}(a,u) = &-\tfrac{1}{2}a(a+\mathrm{i}) + \mathrm{i}\rho\tilde{\nu}(T)ah^{(T)}(a,u)\\ &+ \tfrac{1}{2}\tilde{\nu}(T)^2(h^{(T)})^2(a,t), \quad h^{(T)}(a,0) = 0\end{aligned}$$

This equation can be solved explicitly, as on page 18 of Gatheral (2006). The solution may be written as:

$$h^{(T)}(a,t) = r_-(T)\frac{1 - \mathrm{e}^{-A\tilde{\nu}(T)t}}{1 - (r_-(T)/r_+(T))\mathrm{e}^{-A\tilde{\nu}(T)t}}$$

with:

$$A = \sqrt{a(a+\mathrm{i}) - \rho^2 a^2}, \qquad r_\pm(T) = -\frac{1}{\tilde{\nu}(T)}(\mathrm{i}\rho a \pm A)$$

On the other hand, for a given expiry $T$, a poor man's almost-instantaneous approximation of the rough Heston characteristic function is obtained by approximating the forward variance curve as flat with $\xi_0(u) = v_0(T)$, $u \geqslant 0$. In practice, the obvious choice:

$$v_0(T) = \frac{1}{T}\int_0^T \xi_0(s)\,\mathrm{d}s$$

the fair value of the variance swap, works fine. In that case, (7) becomes:

$$\tilde{\nu}(T) = \sqrt{\frac{3}{2H+2}}\frac{\nu}{\Gamma(H+\frac{3}{2})}\frac{1}{T^{(1/2)-H}} \tag{9}$$

With this choice of forward variance curve, the approximate characteristic function (8) is identical to the characteristic function of the classical Heston model with initial variance $v_0(T)$, mean reversion $\lambda = 0$, correlation $\rho$ and volatility of volatility $\tilde{\nu}(T)$ given by (9). Option prices under rough Heston may therefore be almost instantaneously approximated using any existing implementation of the classical Heston pricing model.

To demonstrate the quality of approximations (7) and (9), in figure 3 we have replotted the rough Heston smiles of figure 2 generated using the Adams scheme together with those generated using the above approximations. We note the smiles generated using the approximate characteristic function (8) with approximation (7) appear to be closer to true rough Heston smiles than those generated by the poor man's Heston approximation, especially close to at-the-money. Nevertheless, the poor man's smiles are surprisingly good. The result is an accurate rough Heston approximation with no quants required.

## Summary

In this article, we have presented the rough Heston model: a particularly tractable rough volatility model with a quasi-closed-form characteristic function analogous to that of the classical Heston model. This characteristic function is in terms of the solution of a fractional Riccati ordinary differential equation, which we show how to solve numerically. Pricing and hedging using the rough Heston model are then straightforward. Calibration to the implied volatility surface is illustrated with two examples; as with other rough volatility models, the shape of the observed surface is faithfully reproduced. Finally, we show how to accurately approximate rough Heston model values by scaling the volatility-of-volatility parameter of the classical Heston model: we call this the poor man's rough Heston model. We thus generate accurate approximations of rough Heston model values with no quants required.

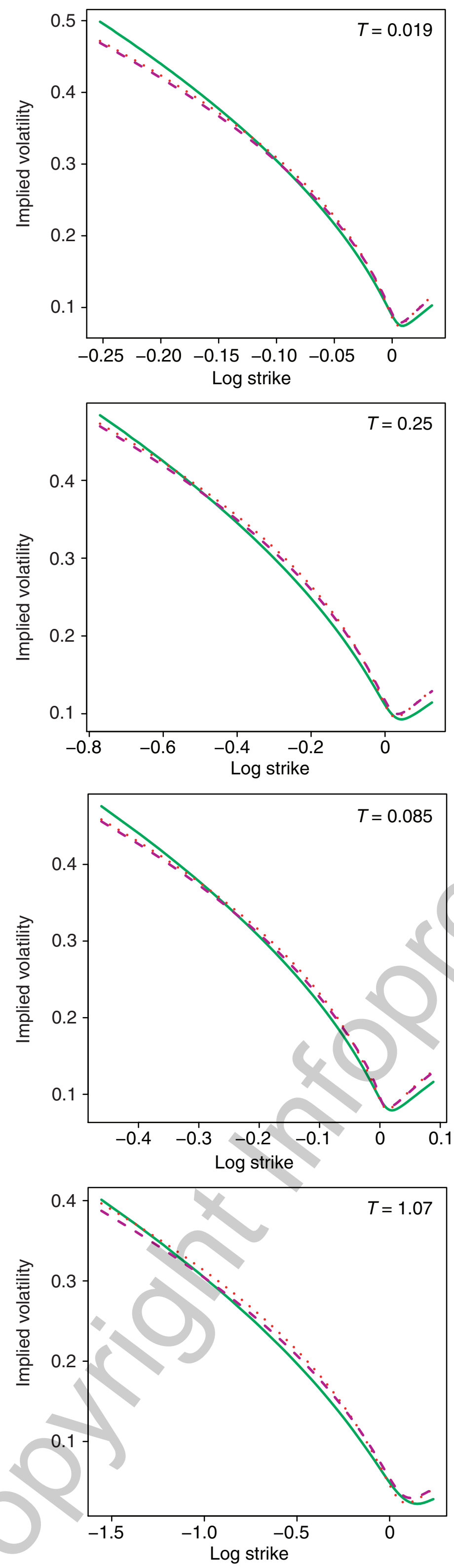
## Appendix A: numerical solution of the fractional Riccati equation

We recall here how to solve fractional ordinary differential equations such as (4). This is needed in order to compute the characteristic function. Specifically, again with $\alpha = H + \frac{1}{2}$, let:

$$D^\alpha h(a,t) = F(a, h(a,t)), \quad h(a,0) = 0 \tag{10}$$

Several schemes for solving (10) numerically are available in the literature. Most of these are based on the idea that (10) implies the following

3 Representative SPX volatility smiles as of May 19, 2017, with time to expiration in years




Green smiles are from the rough Heston model calibrated using the Adams scheme; violet dashed smiles are generated using the approximate rough Heston characteristic function (8); the brown dotted smiles are generated from the poor man's existing Heston model with a scaled volatility-of-volatility parameter

Volterra equation:

$$h(a,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(a,h(a,s))\,\mathrm{d}s \tag{11}$$

One way to solve (11) is to use the classical fractional Adams method presented in Diethelm *et al* (2004). The idea goes as follows. Let us write $g(a,t)=F(a,h(a,t))$. Over a regular discrete time grid with mesh $\Delta$, $0\leqslant t_0\leqslant\cdots\leqslant t_n\leqslant t$, we approximate:

$$h(a,t_{k+1})=\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(a,s)\,\mathrm{d}s$$

by:

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\hat{g}(a,s)\,\mathrm{d}s$$

where:

$$\hat{g}(a,t)=\frac{t_{j+1}-t}{t_{j+1}-t_j}\hat{g}(a,t_j)+\frac{t-t_j}{t_{j+1}-t_j}\hat{g}(a,t_{j+1}),\qquad t\in[t_j,t_{j+1}],\ 0\leqslant j\leqslant k$$

This corresponds to a trapezoidal discretisation and leads to the following scheme:

$$\hat{h}(a,t_{k+1})=\sum_{0\leqslant j\leqslant k}a_{j,k+1}F(a,\hat{h}(a,t_j))+a_{k+1,k+1}F(a,\hat{h}(a,t_{k+1})) \tag{12}$$

with:

$$\left.\begin{aligned}a_{0,k+1}&=\frac{\Delta^\alpha}{\Gamma(\alpha+2)}[k^{\alpha+1}-(k-\alpha)(k+1)^\alpha]\\ a_{j,k+1}&=\frac{\Delta^\alpha}{\Gamma(\alpha+2)}[(k-j+2)^{\alpha+1}+(k-j)^{\alpha+1}\\&\qquad-2(k-j+1)^{\alpha+1}],\quad 1\leqslant j\leqslant k\end{aligned}\right\} \tag{13}$$

and:

$$a_{k+1,k+1}=\frac{(\Delta t)^\alpha}{\Gamma(\alpha+2)}$$

However, as $\hat{h}(a,t_{k+1})$ appears on both sides of (12), this scheme is implicit. Thus, in a first step, we compute a pre-estimation (or 'predictor') of $\hat{h}(a,t_{k+1})$ based on a Riemann sum that we then plug into the trapezoidal quadrature. We define this predictor $\hat{h}^P(a,t_{k+1})$ as follows:

$$\hat{h}^P(a,t_{k+1})=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\tilde{g}(a,s)\,\mathrm{d}s$$

with:

$$\tilde{g}(a,t)=\hat{g}(a,t_j),\quad t\in[t_j,t_{j+1}),\ 0\leqslant j\leqslant k$$

Therefore:

$$\hat{h}^P(a,t_{k+1})=\sum_{0\leqslant j\leqslant k}b_{j,k+1}F(a,\hat{h}(a,t_j))$$

where:

$$b_{j,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha+1)}((k-j+1)^\alpha - (k-j)^\alpha), \quad 0 \leqslant j \leqslant k$$

Thus, the final explicit numerical scheme is given by:

$$\hat{h}(a, t_{k+1}) = \sum_{0 \leqslant j \leqslant k} a_{j,k+1} F(a, \hat{h}(a, t_j)) + a_{k+1,k+1} F(a, \hat{h}^P(a, t_j))$$

where the weights $a_{j,k+1}$ are defined in (13).

## Appendix B: call option prices using Fourier techniques

We now explain how to deal numerically with the fast Fourier transform technique of Carr & Madan (1999) for the computation of call option prices in the specific case of the rough Heston model. Recall that the price at time $t$, $C_t(T, S_0 e^x)$, of the call with expiration time $T > t$ and strike $S_0 e^x$ is related to the characteristic function of the log price $X_T$ through the following expression:

$$C_t(T, S_0 e^x) = \frac{\exp(-\beta x)}{\pi} \int_0^\infty \mathrm{Re}[\psi_t(T, a) \mathrm{e}^{-\mathrm{i}ax}]\, \mathrm{d}a \qquad (14)$$

where:

$$\psi_t(T, a) = \frac{\phi_t(T, a - (\beta+1)\mathrm{i})}{\beta^2 + \beta - a^2 + \mathrm{i}(2\beta+1)a}$$

and $\phi_t$ is defined as in (3). Such a method includes choosing $\beta > 0$ such that:

$$\mathbb{E}_t[S_T^{\beta+1}] < \infty \qquad (15)$$

In El Euch & Rosenbaum (2018), a sufficient condition for finite moments is given. In particular, when $\rho < 0$, the existence of $\beta > 0$ satisfying (15) is guaranteed.

In practice, to apply the fast Fourier transform algorithm, we need to tackle the issue of the infinite upper limit of integration in (14) by looking for $a_{\max} > 0$ such that:

$$\frac{\exp(-\beta x)}{\pi} \int_{a_{\max}}^\infty |\psi_t(T, a)|\, \mathrm{d}a < \varepsilon$$

where $\varepsilon > 0$ is the expected truncation error. In El Euch & Rosenbaum (2018), it is shown that:

$$\mathrm{Re}\bigg[\int_t^T D^\alpha h(a - \mathrm{i}(\beta+1), T-u) \xi_t(u)\, \mathrm{d}u\bigg]$$

is asymptotically dominated as $|a|$ goes to infinity by:

$$-|a| \frac{\sqrt{1-\rho^2}}{\nu \Gamma(1-\alpha)} \int_t^T (T-u)^{-\alpha} \xi_t(u)\, \mathrm{d}u$$

Hence, from (3) it is enough to choose $a_{\max} > 0$ such that:

$$\frac{\exp(-\beta x)}{\pi} S_t^{\beta+1} \times \int_{a_{\max}}^\infty \frac{\exp\Big(-a \frac{\sqrt{1-\rho^2}}{\nu\Gamma(1-\alpha)} \int_t^T (T-u)^{-\alpha} \xi_t(u)\, \mathrm{d}u\Big)}{|\beta^2 + \beta - a^2 + \mathrm{i}(2\beta+1)a|}\, \mathrm{d}a < \varepsilon$$

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