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Addendum

Addendum to Rubtsov and Petrov (2016): "A point-in-time–through-the-cycle approach to rating assignment and probability of default calibration"

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In June 2016, *The Journal of Risk Model Validation* published a paper by Rubtsov and Petrov (2016) called "A point-in-time–through-the-cycle approach to rating assignment and probability of default calibration". On p. 102 of the paper, the authors solved a system of equations (5.7)–(5.9) numerically; these equations are reproduced below as (1)–(3):

$$\mathbb{E}[\Phi^{-1}(d_r)] = \frac{\mu_r - \sqrt{\rho_r}\,\mathbb{E}(\hat{Z})}{\sqrt{1-\rho_r}}, \tag{1}$$

$$\begin{aligned}\mathbb{E}[(\Phi^{-1}(d_r))^2] &= \frac{1}{1-\rho_r}[\mathbb{E}(B_r^2) - 2\sqrt{\rho_r}\,\mathbb{E}(B_r\hat{Z}) + \rho_r\mathbb{E}(\hat{Z}^2)] \\ &= \frac{1}{1-\rho_r}[(\gamma_r + \mu_r^2) - 2\sqrt{\rho_r}\mu_r\mathbb{E}(\hat{Z}) + \rho_r\mathbb{E}(\hat{Z}^2)],\end{aligned} \tag{2}$$

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$$\begin{aligned}\mathbb{E}[(\Phi^{-1}(d_r))^3] &= (1-\rho_r)^{-3/2}[\mathbb{E}(B_r^3) - 3\sqrt{\rho_r}\,\mathbb{E}(B_r^2\hat{Z}) \\ &\qquad + 3\rho_r\mathbb{E}(B_r\hat{Z}^2) - \rho_r^{3/2}\mathbb{E}(\hat{Z}^3)] \\ &= (1-\rho_r)^{-3/2}[(3\mu_r\gamma_r + \mu_r^3) - 3\sqrt{\rho_r}(\mu_r^2+\gamma_r)\mathbb{E}(\hat{Z}) \\ &\qquad + 3\rho_r\mu_r\mathbb{E}(\hat{Z}^2) - \rho_r^{3/2}\mathbb{E}(\hat{Z}^3)]. \end{aligned} \tag{3}$$

Torsten Pyttlik has recently proposed an analytical solution to this system, and we present the details of that solution below. We believe it adds substantial extra value to the original material.

Let $Y_r := \Phi^{-1}(d_r)$ for brevity. The original equations (5.7)–(5.9) then become

$$\mathbb{E}[Y_r] = \frac{\mu_r - \sqrt{\rho_r}\,\mathbb{E}[\hat{Z}]}{\sqrt{1-\rho_r}}, \tag{4}$$

$$\mathbb{E}[Y_r^2] = \frac{1}{1-\rho_r}[\gamma_r + \mu_r^2 - 2\sqrt{\rho_r}\mu_r\mathbb{E}[\hat{Z}] + \rho_r\mathbb{E}[\hat{Z}^2]], \tag{5}$$

$$\begin{aligned}\mathbb{E}[Y_r^3] &= (1-\rho_r)^{-3/2}[\mu_r^3 + 3[\mu_r\gamma_r - \sqrt{\rho_r}(\mu_r^2+\gamma_r)\mathbb{E}[\hat{Z}] \\ &\qquad + \rho_r\mu_r\mathbb{E}[\hat{Z}^2]] - \rho_r^{3/2}\mathbb{E}[\hat{Z}^3]]. \end{aligned} \tag{6}$$

Rearranging (4) gives

$$\mu_r = \sqrt{1-\rho_r}\mathbb{E}[Y_r] + \sqrt{\rho_r}\,\mathbb{E}[\hat{Z}]. \tag{7}$$

Taking the square of (4) and subtracting that from (5) and then rearranging gives us

$$\mathbb{E}[Y_r^2] - \mathbb{E}[Y_r]^2 = \frac{1}{1-\rho_r}[\gamma_r + \rho_r(\mathbb{E}[\hat{Z}^2] - \mathbb{E}[\hat{Z}]^2)], \tag{8}$$

$$\gamma_r = (1-\rho_r)\mathbb{V}[Y_r] - \rho_r\mathbb{V}[\hat{Z}]. \tag{9}$$

Here, we have introduced the variance, defined as

$$\mathbb{V}[X] := \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Note that (9) might result in $\gamma_r < 0$ if $\rho_r > 0$, which is undesirable since γ_r was defined as a variance when the original system of equations was set up. Negative values of ρ_r could therefore be considered, which would require an extensive modification of (1)–(3), using $\sqrt{-\rho_r}$ and changing signs in several places.

Taking the third power of (4) and subtracting this from (6) gives

$$\begin{aligned}\mathbb{E}[Y_r^3] - \mathbb{E}[Y_r]^3 &= (1-\rho_r)^{-3/2}[3[\mu_r\gamma_r - \sqrt{\rho_r}\gamma_r\mathbb{E}[\hat{Z}] + \rho_r\mu_r\mathbb{V}[\hat{Z}]] \\ &\qquad - \rho_r^{3/2}(\mathbb{E}[\hat{Z}^3] - \mathbb{E}[\hat{Z}]^3)]. \end{aligned}$$

Inserting (7) and (9) into the inner square brackets on the right-hand side yields, after rearranging, an expression that is solvable for ρ_r alone:

$$\mathbb{E}[Y_r^3] - 3\mathbb{E}[Y_r]\mathbb{V}[Y_r] - \mathbb{E}[Y_r]^3 = -\left(\frac{\rho_r}{1-\rho_r}\right)^{3/2}[\mathbb{E}[\hat{Z}^3] - 3\mathbb{E}[\hat{Z}]\mathbb{V}[\hat{Z}] - \mathbb{E}[\hat{Z}]^3],$$

$$\rho_r = \frac{1}{[1 + (\mathbb{S}[Y_r]/\mathbb{S}[\hat{Z}])^{-2/3}]}. \tag{10}$$

Here, we have defined

$$\mathbb{S}[X] := \mathbb{E}[X^3] - 3\mathbb{E}[X]\mathbb{V}[X] - \mathbb{E}[X]^3,$$

which is the nonnormalized skewness (to obtain normalized skewness, multiply $\mathbb{S}[X]$ by $\mathbb{V}[X]^{-3/2}$).

After evaluating ρ_r from (10), use (9) and (7) to obtain values for γ_r and μ_r, respectively.

Note that if the distribution of $\hat{Z}$ is symmetrical, ie, $\mathbb{S}[\hat{Z}] = 0$, then (10) has no solution if $\mathbb{S}[Y_r] \neq 0$. There is no unique solution if both $\mathbb{S}[\hat{Z}] = 0$ and $\mathbb{S}[Y_r] = 0$. For the limiting case $\rho_r = 1$, the whole system of equations (4)–(6) would be invalid.

REFERENCES

Rubtsov, M., and Petrov, A. (2016). A point-in-time–through-the-cycle approach to rating assignment and probability of default calibration. *The Journal of Risk Model Validation* **10**(2), 83–112 (http://doi.org/bzcb).