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Transform analysis and asset pricing for diffusion processes: a recursive approach

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Diffusion processes play a major role in continuous-time modeling in economics, particularly in continuous-time finance. In most cases, however, the transition density function of a diffusion process is not available in closed form. Using Feynman–Kac integration, we construct a recursive scheme for the Laplace transform (in time) of the transition density function. This provides a semianalytic and highly accurate solution to a wide range of asset pricing problems. Generalizations of our technique to functionals of non-Gaussian processes are also briefly discussed.

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1 INTRODUCTION

Continuous-time stochastic models have found widespread application in economics. The continuous-time framework now plays a prominent role, not only in financial economics, where continuous-time modeling has become indispensable since the seminal work of Merton in the 1970s, but also in other areas of economics such as macroeconomics and game theory.

The prototypical stochastic model in continuous-time is one in which the dynamics are specified by an Ito stochastic differential equation of the form:

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0 \tag{1.1}$$

Here X_t denotes the variable of interest, μ is the prespecified drift function, σ is the prespecified diffusion and W_t is a standard Brownian motion. Throughout, we restrict our attention to variables in one (spatial) dimension.

A basic object in the study of continuous-time Markov processes is the transition probability (or stochastic kernel). This is the probability of a transition from a point $x_0 \in \mathbb{R}$ at time 0 to an arbitrary interval or set $\Gamma \in \mathbb{R}$ at time t. We denote the transition probability induced by model (1.1) by P_X:

$$P_X(\Gamma, t \mid x_0) = \mathbb{P}[X_t \in \Gamma \mid X_0 = x_0] \tag{1.2}$$

Frequently, P_X will be of the form:

$$P_X(\Gamma, t \mid x_0) = \int_\Gamma p_X(x, t \mid x_0)\,\mathrm{d}x \tag{1.3}$$

where $p_X(x, t \mid x_0)$ denotes the conditional density of $X_t = x$ given $X_0 = x_0$ (also called the transition density function).[1]

A major problem, both probabilistically and statistically, in continuous-time modeling with processes of the form (1.1) is that, with some rare exceptions, the transition density function is not known in closed form. This severely hampers a wide range of asset pricing problems. Traditional financial applications have often been restricted to cases (simple functions μ and σ) with a closed-form analytic solution (Black and Scholes (1973); Vasicek (1977); and Cox *et al* (1985)).

In this paper, we develop a method for obtaining the Laplace transform with respect to time t of the transition density function for arbitrary time-homogeneous diffusion processes of the form (1.1). Our method is based on Feynman–Kac integration techniques, constructing a recursive scheme for the Feynman–Kac integral representation

[1] As is well-known, regularity conditions on μ and σ are needed to ensure the existence of a unique Ito process that solves (1.1) for each starting point x_0, and to ensure that the diffusion X admits a smooth transition density function (see Section 2 for further details).

of its Laplace transform. While various asset pricing problems can already be treated directly from the Laplace transform, the Laplace transform has a one-to-one correspondence with the transition density function, which can be readily derived from it using standard Laplace inversion. Therefore, having the Laplace transform of the transition density function at our disposal is a significant advancement. Due to its semianalytic nature, our method is both accurate and fast, and, moreover, it is easily implemented.

Traditionally, the transition density of a diffusion process is approximated by numerically solving the corresponding Fokker–Planck–Kolmogorov partial differential equation, using, for example, finite difference methods. As is well-known, this may, however, lead to unreliable and nonrobust results due to the fact that the initial condition of this partial differential equation is a Dirac delta function. By contrast, the explicit appearance of a delta function as the initial condition is avoided in our method because it will automatically and analytically be accounted for in the Feynman–Kac formalism. (We return to this issue in Section 3.) Aït-Sahalia (1999, 2002) proposes a Hermite expansion around a Gaussian density function to derive a closed-form approximation to the transition density function for diffusion processes, which is particularly well-suited to short time horizons. Other methods dealing with the approximation of the transition densities, usually inferior in financial applications, include a discrete Euler approximation, a binomial tree approximation and Monte Carlo simulation. For an overview of these methods and a comparative analysis, we refer to Jensen and Poulsen (2002). Duffie *et al* (2000) derive a closed-form expression for a class of transforms (including Laplace) for jump diffusions, thereby allowing for jumps in state variables. They restrict themselves, however, to an affine structure, which is crucial in their setting but which is not required for the validity of our method.

The key advantages of our method when compared with existing methods are therefore:

(1) that it allows for arbitrary (not necessarily affine) time-homogeneous diffusion processes of the form (1.1);

(2) that it is applicable to any time horizon;

(3) that it is of a semianalytic nature, and is hence accurate and fast (especially in problems for which Laplace inversion is not required);

(4) that it is easily implemented.

To the best of our knowledge, no other method satisfies this combination of properties.

The rest of this paper is organized as follows. In Section 2 we describe the connection between the transition density function of a time-homogeneous diffusion process (1.1) and Feynman–Kac integration. In Section 3 we present our recursive scheme

for the Laplace transform of the transition density function. In Section 4 we show how to apply our method to various popular diffusion models and assess the degree of accuracy that our method reaches. Section 5 extends the recursive scheme to include functionals of some non-Gaussian processes and Section 6 contains some concluding remarks.

2 THE TRANSITION DENSITY FUNCTION AND FEYNMAN–KAC INTEGRATION

Various sets of regularity conditions can be imposed on the functions μ and σ. We will adopt the set of assumptions imposed by Aït-Sahalia (2002, Assumptions 1–3), namely, smoothness of the coefficients, nondegeneracy of the diffusion, and appropriate boundary behavior. These assumptions are relatively general and particularly appropriate for applications in financial economics. They ensure the existence of an Ito process, unique in probability, that solves (1.1) for each initial value x_0, and that X admits a smooth transition density function, satisfying the Fokker–Planck–Kolmogorov equation (see (2.4)).

The relation between Feynman–Kac path integration and the transition density function is best understood via the forward Fokker–Planck–Kolmogorov equation. Consider again (1.1). Let us first standardize the diffusion function of X, transforming X into a process Y with unit diffusion by means of the Lamperti transform:

$$Y := \psi(X) = \int^X \frac{\mathrm{d}z}{\sigma(z)} \tag{2.1}$$

From Ito's lemma, we find that:

$$\mathrm{d}Y_t = T(Y_t)\,\mathrm{d}t + \mathrm{d}W_t \tag{2.2}$$

with:

$$T(y) = \frac{\mu(\psi^{-1}(y))}{\sigma(\psi^{-1}(y))} - \frac{1}{2}\frac{\partial\sigma}{\partial x}(\psi^{-1}(y)) \tag{2.3}$$

The transition density function of the unit diffusion process Y satisfies the forward Kolmogorov equation, also called the Fokker–Planck equation:

$$\frac{\partial p_Y(y,t \mid y_0)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}p_Y(y,t \mid y_0) - \frac{\partial}{\partial y}(T(y)p_Y(y,t \mid y_0)) \tag{2.4}$$

with initial condition $p_Y(y,0 \mid y_0) = \delta(y - y_0)$, where, as usual, δ denotes Dirac's delta function.

This linear and parabolic partial differential equation (PDE) in $p_Y(y,t \mid y_0)$ can be transformed into a simpler linear and parabolic PDE in $q_Y(y,t \mid y_0)$ by making

the following substitution:

$$p_Y(y,t \mid y_0) =: q_Y(y,t \mid y_0) \exp\left\{ \int_{y_0}^{y} T(z)\,\mathrm{d}z \right\} \tag{2.5}$$

The exponential factor, upon normalizing, plays a similar role to the Radon–Nikodým derivative in Girsanov's theorem for diffusion processes, paving the way to the construction of an equivalent martingale measure. After a few lines of easy calculus, we obtain:

$$\frac{\partial q_Y(y,t \mid y_0)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2} q_Y(y,t \mid y_0) - \tfrac{1}{2} q_Y(y,t \mid y_0)(T'(y) + T^2(y)) \tag{2.6}$$

with $q_Y(y,0 \mid y_0) = \delta(y - y_0)$. We denote by:

$$V(y) := \tfrac{1}{2}(T'(y) + T^2(y)) \tag{2.7}$$

the potential. It is henceforth assumed to be piecewise continuous and bounded from below (see Kac (1949)).

Then the Feynman–Kac path integral, also called the Feynman–Kac formula, is a probabilistic representation of the solution to the PDE (2.6). That is, q_Y admits the representation:

$$q_Y(y,t \mid y_0) = \mathbb{E}\left[\delta(Y_t - y) \exp\left\{ -\int_0^t V(Y_\tau)\,\mathrm{d}\tau \right\} \;\middle|\; Y_0 = y_0 \right], \quad t \geqslant 0 \tag{2.8}$$

Furthermore, this solution is unique. The Feynman–Kac formula can also be applied to other types of parabolic or elliptic PDEs (see Karatzas and Shreve (1991) for further details). For related applications of the Feynman–Kac formula in financial mathematics, see also Linetsky (1998, 2004b, 2008) and the references therein. For notational convenience, we will write:

$$\mathbb{E}\left[\delta(Y_t - y) \exp\left\{ -\int_0^t V(Y_\tau)\,\mathrm{d}\tau \right\} \;\middle|\; Y_0 = y_0 \right] =: \mathbb{E}^{y,y_0,t}\left[\exp\left\{ -\int_0^t V(Y_\tau)\,\mathrm{d}\tau \right\} \right]$$

where the operator $\mathbb{E}^{y,y_0,t}[\,\cdot\,]$ is shorthand notation for the conditional expectation $\mathbb{E}[\delta(Y_t - y)\cdot \mid Y_0 = y_0]$ on the unit-diffusion process Y_t, having positions y_0 at time 0 and y at time t (a "bridge"). Furthermore, we write $q_Y(y,t \mid y_0) =: k(y,t \mid y_0)$ as common notation for kernel. To obtain the original density $p_X(x,t \mid x_0)$, we use (2.5) and apply the Jacobian formula for the change of density:

$$p_X(x,t \mid x_0) = \frac{1}{\sigma(x)} p_Y(\psi(x),t \mid \psi(x_0))$$

We may define the Laplace transform (with respect to time t):

$$\rho^s(y,y_0) := \int_0^\infty \mathrm{e}^{-st} k(y,t \mid y_0)\,\mathrm{d}t, \quad s > 0 \tag{2.9}$$

A sufficient condition for the existence of ρ^s is that the potential V is nonnegative because in that case k is bounded and nonincreasing in t. As we will see (see Remark 3.2), we can assume without loss of generality that the potential is nonnegative, and we therefore impose this assumption henceforth. Then, with $\lim_{t\to\infty} \mathrm{e}^{-st}k(y,t \mid y_0) = 0$:

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\rho^s(y,y_0) - (s + V(y))\rho^s(y,y_0) + \delta(y - y_0) = 0 \tag{2.10}$$

with boundary conditions:

$$\rho^s(y,y_0) \to 0 \quad \text{as } y \to \pm\infty$$

and which is nondifferentiable (singular) at $y = y_0$:

$$\rho^{s\prime}(y_0+,y_0) - \rho^{s\prime}(y_0-,y_0) = -2$$

(cf, Kac (1949, 1951)). The probabilistic representation of the solution to the ordinary differential equation (ODE) (2.10) is sometimes referred to as the Kac formula. Kac (1949) proved the existence and uniqueness of a fundamental solution (Green's function) to the ODE (2.10). In quantum physics the Feynman–Kac path integral is an instrument that is widely used for calculating energy state and wave functions. A Feynman–Kac path integral provides an explicit expression of Green's function corresponding to the Schrödinger equation (see Schulman (1981) and Simon (2005) for details).

We easily verify analytically that the transition density function for continuous-time Markov processes satisfies the composition rule:

$$p_X(x,t \mid x_0) = \int_{\mathbb{R}} p_X(y,\tau \mid x_0)\, p_X(x,t-\tau \mid y)\,\mathrm{d}y, \quad 0 < \tau < t \tag{2.11}$$

It is referred to as the Chapman–Kolmogorov equation and states that the conditional density of $X_t = x$ given $X_0 = x_0$ can be decomposed into a conditional density of $X_\tau = y$ given $X_0 = x_0$ and a conditional density of $X_t = x$ given (only) $X_\tau = y$ (with $X_0 = x_0$ being irrelevant), where τ is an arbitrary time point between 0 and t.

3 THE RECURSION RELATION FOR THE FEYNMAN–KAC INTEGRAL

Throughout this section, we let the diffusion function of X be unity; whenever needed, the Lamperti transform discussed in the previous section can be preapplied to obtain a diffusion process with unit diffusion. We additively decompose the corresponding potential V given by (2.7) into two parts: $V(x) = V_1(x) + V_2(x)$, with V_1 and

V_2 specified below. We choose the additive decomposition such that the Laplace transform of the probabilistic representation (Feynman–Kac formula) related to V_1:

$$\int_0^\infty \mathrm{e}^{-st}\mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t V_1(X_\tau)\,\mathrm{d}\tau\right\}\right]\mathrm{d}t =: \int_0^\infty \mathrm{e}^{-st}k_1(x,t\mid x_0)\,\mathrm{d}t$$
$$=: \rho_1^s(x,x_0)$$

can be explicitly evaluated. We then consider:

$$k_{1,2}(x,t\mid x_0) := \mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t V_1(X_\tau)\,\mathrm{d}\tau - \int_0^t V_2(X_\tau)\,\mathrm{d}\tau\right\}\right] \tag{3.1}$$

and its Laplace transform:

$$\rho_{1,2}^s(x,x_0) := \int_0^\infty \mathrm{e}^{-st}k_{1,2}(x,t\mid x_0)\,\mathrm{d}t$$

The standing assumption that V is nonnegative, and the fact that V_1 and V_2, specified below, are also nonnegative, ensures that k_1, $k_{1,2}$ and their Laplace transforms are well-defined. We derive a recursive scheme for the calculation of $\rho_{1,2}^s$.

To this end, we first expand $\exp\{-\int_0^t V_2(x_\tau)\,\mathrm{d}\tau\}$:

$$\exp\left\{-\int_0^t V_2(x_\tau)\,\mathrm{d}\tau\right\} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\left[\int_0^t V_2(x_\tau)\,\mathrm{d}\tau\right]^n$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^t \mathrm{d}\tau_1\cdots\int_0^t \mathrm{d}\tau_n V_2(x_{\tau_1})\cdots V_2(x_{\tau_n})$$

where the $n=0$ term in the summation yields 1 by convention. Due to symmetry, the right-hand side of the expression above reduces to:

$$\sum_{n=0}^\infty (-1)^n \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1}\cdots\int_0^{\tau_2} \mathrm{d}\tau_1 V_2(x_{\tau_n})\cdots V_2(x_{\tau_1})$$

Below, V_2 will be specified as a weighted delta function potential. Hence, in that case, substituting the above expansion of $\exp\{-\int_0^t V_2(x_\tau)\,\mathrm{d}\tau\}$ in (3.1) and using the Chapman–Kolmogorov property, one obtains, in the spirit of Goovaerts *et al* (1973) and Goovaerts and Broeckx (1985):

$$k_{1,2}(x,t\mid x_0)$$
$$= \sum_{n=0}^\infty (-1)^n \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1}\cdots\int_0^{\tau_2} \mathrm{d}\tau_1$$
$$\times \int_{-\infty}^\infty \mathrm{d}x_n\cdots\int_{-\infty}^\infty \mathrm{d}x_1 k_1(x,t-\tau_n\mid x_n)V_2(x_n)\cdots V_2(x_1)k_1(x_1,\tau_1\mid x_0)$$

Taking the Laplace transform of the kernel $k_{1,2}$ with respect to time t yields:

$$\rho^s_{1,2}(x,x_0) = \sum_{n=0}^{\infty}(-1)^n \int_{-\infty}^{\infty} \mathrm{d}x_n \cdots \int_{-\infty}^{\infty} \mathrm{d}x_1 \rho^s_1(x,x_n)V_2(x_n)\cdots V_2(x_1)\rho^s_1(x_1,x_0) \tag{3.2}$$

For an arbitrary (piecewise continuous and nonnegative) potential V, we introduce the following integral representation, for each continuity point of V:

$$\begin{aligned} V(x) &= \int_{-\infty}^{\infty} V(a)\delta(x-a)\,\mathrm{d}a \\ &= \lim_{l\to\infty}\sum_{j=0}^{l}(a_{j+1,l}-a_{j,l})V(a_{j,l})\delta(x-a_{j,l}) \end{aligned} \tag{3.3}$$

for a sequence of partitions R_l given by $R_l = \{a_{0,l}, a_{1,l}, \ldots, a_{l,l}\}$ satisfying $a_{0,l} < a_{1,l} < \cdots < a_{l,l}$, with:

$$\begin{gathered} \lim_{l\to\infty}\max_{0\leq j\leq l-1}|a_{j+1,l}-a_{j,l}| = 0 \\ \lim_{l\to\infty} a_{l,l} = \infty \\ \lim_{l\to\infty} a_{0,l} = -\infty \end{gathered}$$

We note that, because the delta function is not a "proper" function, the limit relation on the right-hand side of (3.3) needs to be understood in measure-theoretic sense.

We fix a sufficiently fine partition R_l and henceforth suppress the second index l. To set up the recursion, we then start with a linear combination of $m+1$ delta function potentials:

$$V^{(m)}(x) := \sum_{j=0}^{m}(a_{j+1}-a_j)V(a_j)\delta(x-a_j)$$

As alluded to above, the Feynman–Kac formalism automatically allows for the calculation of (weighted combinations of) delta function potentials (see also Remark 3.1).

Now suppose we know the $(m+1)$th recursion term, $\rho^{s,(m)}(x,x_0)$ for some $V^{(m)}(x)$. We then set $V_1(x) \equiv V^{(m)}(x)$ (the choice of V_1 is adapted at each step of the recursion). Next, adding an additional delta function:

$$(a_{m+2}-a_{m+1})V(a_{m+1})\delta(x-a_{m+1})$$

and recalling (3.2), we obtain the $(m+2)$th recursion term as:

$$\begin{aligned}
&\rho^{s,(m+1)}(x,x_0)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\int_{-\infty}^{\infty}\mathrm{d}x_n\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\\
&\qquad\times\rho^{s,(m)}(x,x_n)(a_{m+2}-a_{m+1})V(a_{m+1})\delta(x_n-a_{m+1})\\
&\qquad\times\cdots(a_{m+2}-a_{m+1})V(a_{m+1})\delta(x_1-a_{m+1})\rho^{s,(m)}(x_1,x_0)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n(a_{m+2}-a_{m+1})^nV^n(a_{m+1})\\
&\qquad\times\rho^{s,(m)}(x,a_{m+1})(\rho^{s,(m)}(a_{m+1},a_{m+1}))^{n-1}\rho^{s,(m)}(a_{m+1},x_0)\\
&\quad=\rho^{s,(m)}(x,x_0)-\frac{\rho^{s,(m)}(x,a_{m+1})\rho^{s,(m)}(a_{m+1},x_0)V(a_{m+1})(a_{m+2}-a_{m+1})}{1+\rho^{s,(m)}(a_{m+1},a_{m+1})V(a_{m+1})(a_{m+2}-a_{m+1})}
\end{aligned}\tag{3.4}$$

where the last equality holds by virtue of the geometric series. (With $a_{m+2}-a_{m+1}$ sufficiently small, that is, with a sufficiently fine partition R, $\rho^{s,(m)}(a_{m+1},a_{m+1})\times V(a_{m+1})(a_{m+2}-a_{m+1})$ is smaller than 1 for each $m=0,\ldots,N-1$, which ensures that the geometric series converges. Here, N is the number of steps in the recursion.) As $l\to\infty$, $\{\rho^{s,(m)}(x,x_0)\}_{m=0}^{l}$ is convergent to $\rho_{1,2}^{s}(x,x_0)$ for all complex s and consequently for all real nonnegative s. The initial value of the recursion is obtained by setting $V_1=0$. Below we summarize the key elements of the recursion.

Recursive scheme for Laplace transformed transition densities

Input. Potential V, Laplace transform parameter s, number of recursions N, partition R.

Initial value.

(1) $\rho^{s,(0)}(x,x_0)=(1/\sqrt{2s})\mathrm{e}^{-\sqrt{2s}|x-x_0|}$ provided the domain of the diffusion is $(-\infty,\infty)$.

(2) $\rho^{s,(0)}(x,x_0)=(\mathrm{e}^{-\sqrt{2s}|x-x_0|}-\mathrm{e}^{-\sqrt{2s}|x+x_0|})/\sqrt{2s}$ if the domain of the diffusion is $(0,\infty)$.

Recursion: from $m=0$ to $N-1$.

$$\begin{aligned}
&\rho^{s,(m+1)}(x,x_0)\\
&\quad=\rho^{s,(m)}(x,x_0)-\frac{\rho^{s,(m)}(x,a_{m+1})\rho^{s,(m)}(a_{m+1},x_0)V(a_{m+1})(a_{m+2}-a_{m+1})}{1+\rho^{s,(m)}(a_{m+1},a_{m+1})V(a_{m+1})(a_{m+2}-a_{m+1})}
\end{aligned}$$

REMARK 3.1 The method developed above constructs analytically exact results for combinations (weighted sums) of delta functions, which reflect Dirichlet boundaries in Feynman–Kac formalism. To verify, consider the PDE (see (2.6)):

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}k(x,t \mid x_0) - V(x)k(x,t \mid x_0) = \frac{\partial}{\partial t}k(x,t \mid x_0)$$

with solution $k(x,t \mid x_0)$, the probabilistic representation of which can be expressed as a Feynman–Kac path integral. Taking the Laplace transform with respect to time t on both sides, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial x^2}\rho^s(x,x_0) - V(x)\rho^s(x,x_0)&\\ = \lim_{t\to\infty} \mathrm{e}^{-st}k(x,t \mid x_0) - \lim_{t\to 0} \mathrm{e}^{-st}k(x,t \mid x_0) + s\rho^s(x,x_0)&\end{aligned}$$

which leads to:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\rho^s(x,x_0) - (s+V(x))\rho^s(x,x_0) = -\delta(x-x_0) \tag{3.5}$$

For any partition:

$$R_l = \{a_{0,l}, a_{1,l}, \ldots, a_{l,l}\} \quad \text{with } a_{0,l} < a_{1,l} < \cdots < a_{l,l}$$

we let $\Delta a_{j,l} := a_{j+1,l} - a_{j,l}$ be the width of the subinterval j. Eventually we let $\max_{0\leqslant j\leqslant l-1} |a_{j+1,l} - a_{j,l}|$ converge to 0 as l converges to infinity and let:

$$\lim_{l\to\infty} a_{l,l} = \infty \quad \text{and} \quad \lim_{l\to\infty} a_{0,l} = -\infty$$

By the arbitrariness of the potential, the proposed recursion, then, gives us the analytically exact solution of the PDE for each partition R_l (suppressing again the second index l):

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\rho^{s,(l)}(x,x_0) - \left(\sum_{j=0}^{l} V(a_j)\delta(x-a_j)\Delta a_j + s\right)\rho^{s,(l)}(x,x_0) = -\delta(x-x_0)$$

with fundamental (ie, analytic representation of) solution:

$$\rho^{s,(l)}(x,x_0) = \int_0^\infty \mathrm{e}^{-st}\mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t \sum_{j=0}^{l} V(a_j)\delta(X_\tau - a_j)\Delta a_j \,\mathrm{d}\tau\right\}\right]\mathrm{d}t \tag{3.6}$$

As soon as the series development (3.4) converges, which can, under the standing assumptions on V, always be achieved by picking a sufficiently fine partition, the analytically exact result for $\rho^{s,(l)}(x, x_0)$ is obtained, for the specific potential consisting of a weighted sum of delta function potentials. We emphasize that this is not an approximation: it is an analytically exact result for each R_l. This is a key feature of our approach: in each step of the recursion, an additional delta function is added, which allows an analytically exact evaluation, and our method thus avoids any numerical integration. When the limit for $l \to \infty$ is taken, we get the desired result $\rho^s_{1,2}(x, x_0)$.

REMARK 3.2 When deriving our recursive scheme, we have assumed that $V(x) \geqslant 0$. In the case where $V_{\min} := \inf_x V(x)$ satisfies $M < V_{\min} < 0$ for some real M (ie, V is merely bounded from below), we can rewrite our problem by considering:

$$\mathrm{e}^{-tV_{\min}}\tilde{k}(x, t \mid x_0) := \mathrm{e}^{-tV_{\min}}\mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t \tilde{V}(X_\tau)\,\mathrm{d}\tau\right\}\right] \tag{3.7}$$

with $\tilde{V}(x) := V(x) - V_{\min}$, and restrict attention to the computation of $\tilde{k}$. This means that if V is negative somewhere but bounded from below, a simple change of potential from V to $\tilde{V}$ accomplishes that the potential inside the Feynman–Kac integral is nonnegative as required. We will discuss this particular situation further in the double-well potential example (see Section 4.4).

REMARK 3.3 In Linetsky (2004b), an additional nonnegative and continuous function, referred to as the killing rate, is considered in the argument of the exponential function in (2.8), hence (3.1). This is relevant in interest rate and credit risk applications. Mathematically, the killing rate can simply be absorbed by the potential and, hence, can be dealt with exactly as above.

4 EXAMPLES

In this section, we study the accuracy of the recursive scheme introduced in the previous section numerically. To this end, we first present some examples in which closed-form expressions for the transition density functions are, in fact, available. We find that the recursion results are highly accurate, both before and after we carry out the real Laplace inversion,[2] when compared with the closed-form results. This is true even for a reasonably limited number of recursions. Moreover, we find that the recursive scheme provides accurate results not only for short time horizons (eg, $t = 1$ year) but also for longer time horizons (eg, $t = 10$ years). Next, we consider the double-well potential model and the Chan–Karolyi–Longstaff–Sanders (CKLS) model (see Chan *et al* (1992)), which nests many well-known interest rate models

[2] Details of the method used to conduct Laplace inversion are given in Appendix A.

as special cases. Under both models, no closed-form expressions for the transition density functions exist in the literature.

The probability mass of a transition density approximation is used as one of the measures of accuracy in this section. Whenever accurate, the approximation should conserve probability and the probability mass of the approximation should therefore be approximately equal to one.

4.1 The Vasicek model

Vasicek (1977) proposed that the instantaneous spot interest rate evolves as an Ornstein–Uhlenbeck process with constant parameters:

$$\mathrm{d}X_t = \kappa(\alpha - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

where κ, α, σ are nonnegative constants. In this case, the transition density function can be worked out explicitly as:

$$\begin{aligned} p_X(x,t \mid x_0) &= \frac{1}{\sigma}\exp\left\{\frac{\kappa\alpha}{\sigma^2}(x-x_0) - \frac{\kappa}{2\sigma^2}(x^2-x_0^2) - \frac{\kappa^2\alpha^2}{2\sigma^2}t + \frac{\kappa}{2}t\right\} \\ &\quad \times \mathbb{E}^{x/\sigma,x_0/\sigma,t}\left[\exp\left\{-\tfrac{1}{2}\int_0^t (\kappa^2 X_\tau^2 - \frac{2\kappa^2\alpha X_\tau}{\sigma})\,\mathrm{d}\tau\right\}\right] \\ &= \sqrt{\frac{\kappa}{\pi\sigma^2(1-\mathrm{e}^{-2\kappa t})}}\exp\left\{-\frac{\kappa(x-\alpha-(x_0-\alpha)\mathrm{e}^{-\kappa t})^2}{\sigma^2(1-\mathrm{e}^{-2\kappa t})}\right\} \end{aligned}$$

In the following, we compare the recursive scheme and the closed-form results for the quadratic potential case in which $V(x) = x^2$. As can be verified from the results in Section 2, a quadratic potential corresponds to the dynamics of (a special case of) the Vasicek model ($\kappa = \sqrt{2}, \alpha = 0, \sigma = 1$).[3] The Laplace transform of the transition density can be obtained by numerically evaluating (2.9) for each (x, x_0), where:

$$\begin{aligned} k(x,t \mid x_0) &= \mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t X_\tau^2\,\mathrm{d}\tau\right\}\right] \\ &= \frac{\exp\{-(1/\sqrt{2}\sinh(\sqrt{2}t))[\cosh(\sqrt{2}t)(x^2+x_0^2) - 2xx_0]\}}{\sqrt{\sqrt{2}\pi\,\sinh(\sqrt{2}t)}} \end{aligned} \tag{4.1}$$

[3] A list of the functions $V(x)$ and $T(x)$ for some popular diffusion models is provided in Table 1 on the facing page and Table 2 on page 60. Schulman (1981) provides expressions for the exact Green function of the action along the classical path with imaginary arguments.

TABLE 1 Potentials in the path integral representation of the transition density for popular diffusion processes.

Model	Diffusion equation	Domain	Potential $V(x)$	$T(x)$
Wiener	$\mathrm{d}X_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, σ positive constant	$(-\infty, \infty)$	$\dfrac{\mu^2}{2\sigma^2}$	$\dfrac{\mu}{\sigma}$
Geometric Wiener	$\mathrm{d}X_t = (\mu + \frac{1}{2}\sigma^2)X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t$, σ positive constant	$(0, \infty)$	$\dfrac{\mu^2}{2\sigma^2}$	$\dfrac{\mu}{\sigma}$
Vasicek	$\mathrm{d}X_t = \kappa(\alpha - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, κ, α, σ positive constants	$(-\infty, \infty)$	$\frac{1}{2}\kappa^2 x^2 - \dfrac{\kappa^2\alpha}{\sigma}x + \dfrac{\kappa^2\alpha^2}{2\sigma^2} - \frac{1}{2}\kappa$	$\dfrac{\kappa\alpha}{\sigma} - \kappa x$
CIR	$\mathrm{d}X_t = \kappa(\alpha - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t$, $\kappa\,\alpha$, σ positive constants, $2\kappa\alpha \geqslant \sigma^2$	$(0, \infty)$	$\frac{1}{8}\kappa^2 x^2 + \left(\dfrac{2\kappa^2\alpha^2}{\sigma^4} - \dfrac{2\kappa\alpha}{\sigma^2} + \dfrac{3}{8}\right)\dfrac{1}{x^2} - \dfrac{\kappa^2\alpha}{\sigma^2}$	$\left(\dfrac{2\kappa\alpha}{\sigma^2} - \dfrac{1}{2}\right)\dfrac{1}{x} - \frac{1}{2}\kappa x$
Adapted geometric Wiener	$\mathrm{d}X_t = ((\delta + \frac{1}{2}\sigma^2)X_t - 1)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t$	$(0, \infty)$	$\dfrac{1}{2}\left(1 - \dfrac{2\delta}{\sigma^2}\right)\mathrm{e}^{-\sigma x} + \dfrac{1}{2\sigma^2}\mathrm{e}^{-2\sigma x} + \dfrac{\delta^2}{2\sigma^2}$	$\dfrac{\delta}{\sigma} - \dfrac{1}{\sigma}\mathrm{e}^{-\sigma x}$
Bessel with drift	$\mathrm{d}X_t = \left(\dfrac{1}{X_t} - 2\right)\mathrm{d}t + \mathrm{d}W_t$	$(0, \infty)$	$2 - \dfrac{2}{x}$	$\dfrac{1}{x} - 2$

Explicit transition density available.

TABLE 2 Potentials in the path integral representation of the transition density for popular diffusion processes.

Model	Diffusion equation	Domain	Potential $V(x)$	$T(x)$
CKLS	$dX_t = \kappa(\alpha - X_t)\,dt + \sigma X_t^{3/2}\,dW_t$, κ, α, σ positive constants	$(0, \infty)$	$\frac{1}{8}(\frac{3}{x^2} + (\kappa^2 - 3\kappa\alpha\sigma^2)x^2 - \frac{1}{2}\kappa^2\alpha\sigma^2x^4 + \frac{1}{16}\kappa^2\alpha^2\sigma^4x^6) + \kappa$	$-\frac{1}{8}\kappa\alpha\sigma^2x^3 + \frac{3}{2x} + \frac{1}{2}\kappa x$
Double-well potential	$dX_t = (X_t - X_t^3)\,dt + dW_t$	$(-\infty, \infty)$	$\frac{1}{2}(x^6 - 2x^4 - 2x^2 + 1)$	$x - x^3$

Explicit transition density not available.

(see Feynman and Hibbs (1965)). Since, for $t \to 0$, the integrand of (2.9) behaves like the delta function $\delta(x - x_0)$, we (have to) rewrite it as:

$$\rho^s(x, x_0) = \frac{\exp\{-\sqrt{2s}|x - x_0|\}}{\sqrt{2s}} + \int_0^\infty \mathrm{e}^{-st}\left[\frac{\exp\{-(1/\sqrt{2}\sinh(\sqrt{2}t))[\cosh(\sqrt{2}t)(x^2 + x_0^2) - 2xx_0]\}}{\sqrt{\sqrt{2}\pi\sinh(\sqrt{2}t)}} - \frac{\exp\{-(x - x_0)^2/2t\}}{\sqrt{2\pi t}}\right]\mathrm{d}t \tag{4.2}$$

where the first term is derived from:

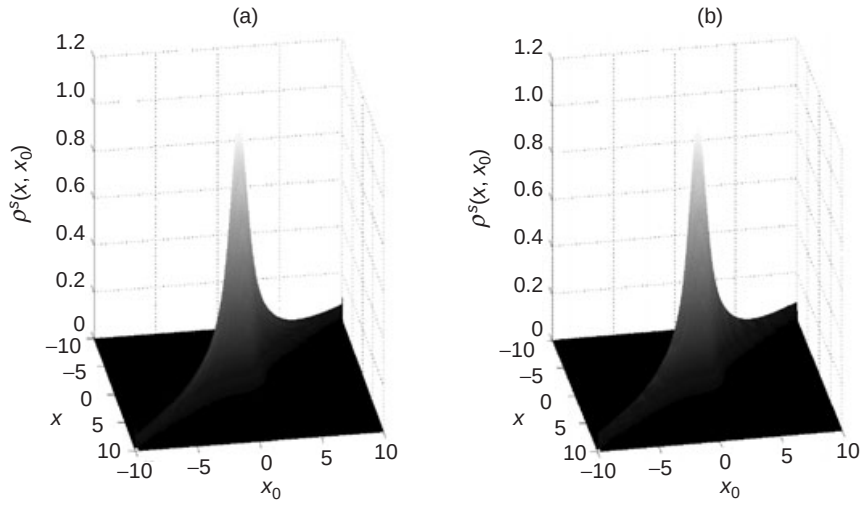
$$\int_0^\infty \mathrm{e}^{-st}\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(x - x_0)^2}{2t}\right\}\mathrm{d}t = \frac{\exp\{-\sqrt{2s}|x - x_0|\}}{\sqrt{2s}}$$

cf, the initial value of the recursive scheme.

We take the parameter of the Laplace transform equal to $s = 0.1$, set the number of recursions to $N = 400$ and compare the recursion results with their closed-form (but numerically evaluated) counterparts. Figure 1 on the next page illustrates the surfaces of the recursion results and their closed-from counterparts for the Laplace transformed transition densities, and the results given different starting values x_0 are in Figure 2 on page 63. In Table 3 on page 64, we compare some sample values of our recursion results with the closed-form results. Since the matrix of the transition density is symmetric, we only pick four positive initial values x_0 to demonstrate the accuracy of the recursion. Moreover, in order to have a detailed view on the performance of the recursion, Table 4 on page 65 contains the results for different values of s. The recursion results prove to be highly accurate for different choices of s, and the computation time for the recursion is much smaller than for the closed-form results. This is due to the fact that we have to perform a numerical integration from zero to infinity to obtain the closed-form results, doing this for every combination (x, x_0).

After examining the accuracy and convergence of the recursive scheme in the Laplace transform domain, we now use the algorithm described in Appendix A to invert the Laplace transform. As mentioned in Appendix A, there is a trade-off between the accuracy and efficiency when selecting the number of the polynomial terms M in the approximation. Additionally, in the process of discretization, we also have to make a compromise between the discretization error and the computer's roundoff errors stemming from the computer's rounding after each calculation, and the latter increase as the former errors are reduced. In this example, we will use $N = 100$,

FIGURE 1 Three-dimensional surface of Laplace transformed transition densities under the Vasicek model.



Surfaces of (a) the recursion results and (b) the exact (closed-form) calculation of the Laplace transformed transition densities $\rho^s(x, x_0)$ with quadratic potential $V(x) = x^2$, given $s = 0.1$, $a_0 = -10$, $N = 400$. "Recursion" refers to the recursive scheme, and "exact calculation" refers to the results obtained from Equation (4.2). The full recursion (on the entire grid) takes around 1.0 seconds, whereas the exact calculations take around 470 seconds due to the involvement of a numerical integration from 0 to ∞ for each (x, x_0).

$M = 8$, $a_0 = 5$. In part (a) of Figure 3 on page 65, we plot the transition probability densities when $t = 1$ and the recursion results fit the closed-form results almost perfectly. This also works for the longer duration transition densities (eg, $t = 10$), as we can see from part (b) of Figure 3 on page 65.

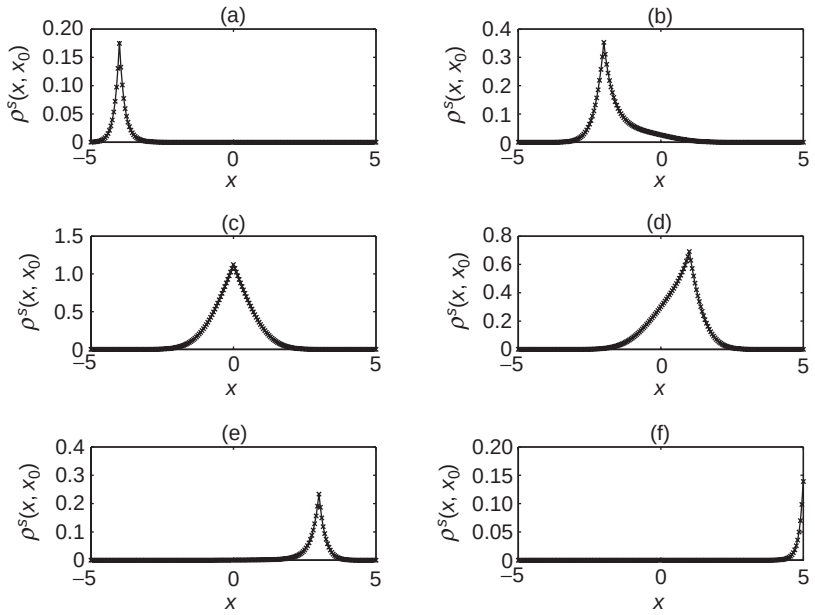
4.2 An annuity certain with exponential time distribution

Considering an annuity certain with stochastic interest governed by a Brownian motion, we may be interested in determining the Laplace transform of:

$$Z_{\mathbb{T}} := \int_0^{\mathbb{T}} \mathrm{e}^{-\delta t - X_t}\, \mathrm{d}t$$

with an exponentially distributed time $\mathbb{T}$, and where δ is the risk-free rate and X_t is a Wiener process (see Yor (2001) and Linetsky (2004a, 2006) for related applications). We then have to calculate $\mathbb{E}_{\mathbb{T}}[\mathbb{E}^{x,x_0,\mathbb{T}}[\mathrm{e}^{-uZ_{\mathbb{T}}}]]$ in order to determine the Laplace transform of $Z_{\mathbb{T}}$. Analytic results for this situation are available and are compared with our algorithm.

FIGURE 2 Laplace transformed transition densities with given initial status under the Vasicek model.



Comparison between the recursion results and the exact (closed-form) calculation (4.2) of the Laplace transformed transition densities $\rho^s(x, x_0)$ with quadratic potential $V(x) = x^2$, given (a) $x_0 = -4$, (b) $x_0 = -2$, (c) $x_0 = 0$, (d) $x_0 = 1$, (e) $x_0 = 3$ and (f) $x_0 = 5$. Solid line: exact. Crosses: recursion.

De Schepper *et al* (1992) developed a method for calculating the moment generating function of the annuity certain $Z_{\mathbb{T}}$ given that the underlying interest rates follow a Wiener process. They derived the following expression for the double Laplace transform of the annuity certain with respect to t and x, when time is exponentially distributed:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-ux} \int_0^\infty s\mathrm{e}^{-st} f_{Z_{\mathbb{T}}}(x)\,\mathrm{d}t\,\mathrm{d}x \\ = 4sI_{2\sqrt{2s+\delta^2}}(2\sqrt{2u}) \int_{-\infty}^0 \mathrm{e}^{\delta x_t} K_{2\sqrt{2s+\delta^2}}(2\sqrt{2u}\mathrm{e}^{-x_t/2})\,\mathrm{d}x_t \\ + 4sK_{2\sqrt{2s+\delta^2}}(2\sqrt{2u}) \int_0^\infty \mathrm{e}^{\delta x_t} I_{2\sqrt{2s+\delta^2}}(2\sqrt{2u}\mathrm{e}^{-x_t/2})\,\mathrm{d}x_t\end{aligned}$$

where $f_{Z_{\mathbb{T}}}(x)$ is the probability density function of the annuity certain $Z_{\mathbb{T}}$, and $I_\nu(z)$ and $K_\nu(z)$ are the modified Bessel functions of the first and second kind of order ν, respectively.

TABLE 3 Laplace transformed transition densities with given initial status under the Vasicek model.

	$x=-3$	$x=-2$	$x=-1$	$x=-0.5$	$x=0$	$x=0.5$	$x=1$	$x=2$	$x=3$
$x_0=0$	0.00062	0.02606	0.30091	0.65524	1.1226	0.65524	0.30091	0.02606	0.00062
	0.00063	*0.02608*	*0.30097*	*0.65524*	*1.1225*	*0.65524*	*0.30097*	*0.02608*	*0.00063*
	$x=-2.5$	$x=-2$	$x=-1$	$x=-0.5$	$x=0$	$x=0.5$	$x=1$	$x=2$	$x=3$
$x_0=2$	0.00011	0.00060	0.00699	0.01521	0.02606	0.03876	0.05982	0.3529	0.00833
	0.00013	*0.00062*	*0.00700*	*0.01523*	*0.02608*	*0.03878*	*0.05987*	*0.3541*	*0.00834*
	$x=2.5$	$x=3$	$x=3.5$	$x=3.7$	$x=3.85$	$x=4$	$x=4.15$	$x=4.5$	$x=5$
$x_0=4$	0.00022	0.00143	0.01319	0.03550	0.07748	0.17461	0.07227	0.00818	0.00027
	0.00024	*0.00144*	*0.01322*	*0.03567*	*0.07796*	*0.17769*	*0.07274*	*0.00820*	*0.00029*
	$x=5.1$	$x=5.3$	$x=5.5$	$x=5.7$	$x=5.9$	$x=6.1$	$x=6.3$	$x=6.5$	$x=6.7$
$x_0=6$	0.00011	0.00047	0.00211	0.01002	0.050322	0.04878	0.00840	0.00137	0.00021
	0.00013	*0.00048*	*0.00211*	*0.01009*	*0.05133*	*0.04979*	*0.00845*	*0.00137*	*0.00022*

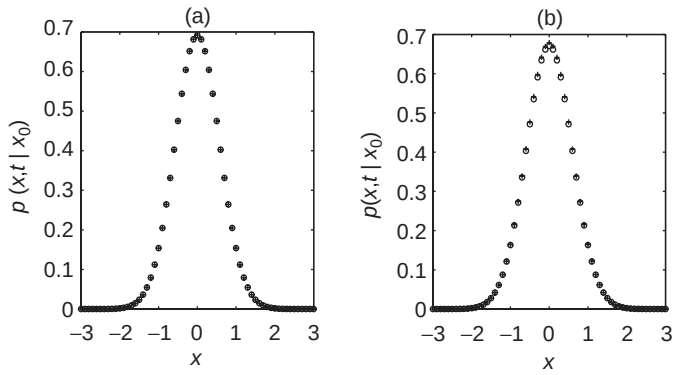
Recursion results of the Laplace transformed transition densities compared with the exact (closed-form) results (4.2) with quadratic potential $V(x)=x^2$, given $s=0.1$ and $x_0=0,2,4,6$. The negative part of x_0 is ignored since the matrix of the Laplace transformed densities is symmetric. The exact results are given in italics.

TABLE 4 Laplace transformed transition densities with different combinations of starting and end values under the Vasicek model.

	$x_0 = 0$, $x = 1$	$x_0 = 2$, $x = 0$	$x_0 = 4$, $x = 3.5$	$x_0 = 6$, $x = 6.5$
$s = 0.1$	0.30091	0.02606	0.01319	0.00137
	0.30097	*0.02608*	*0.01322*	*0.00137*
$s = 0.3$	0.22493	0.01807	0.01285	0.00135
	0.22497	*0.01807*	*0.01286*	*0.00134*
$s = 2$	0.17543	0.01309	0.01252	0.00133
	0.17547	*0.01310*	*0.01254*	*0.00132*
$s = 1$	0.10569	0.00662	0.01175	0.00129
	0.10572	*0.00662*	*0.01177*	*0.00127*
$s = 12$	0.07036	0.00373	0.01104	0.00124
	0.07038	*0.00373*	*0.01106*	*0.00123*
$s = 2$	0.04978	0.00225	0.01039	0.00120
	0.04980	*0.00225*	*0.01041*	*0.00119*

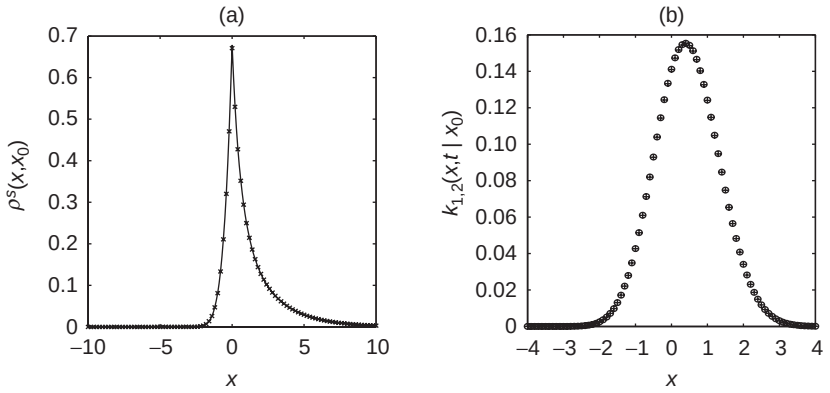
Recursion results of the Laplace transformed transition densities compared with the exact (closed-form) results (4.2) with quadratic potential $V(x) = x^2$, for several combinations of x_0 and x, given $s = 0.1, 0.3, 0.5, 1, 1.5, 2$. The exact results are given in italics.

FIGURE 3 Transition densities under the Vasicek model.



Transition density $p(x, t \mid x_0)$ obtained by exact (closed-form) results and the recursive scheme after the real Laplace inversion for (a) short time horizon ($t = 1$) and (b) long time horizon ($t = 10$) given $V(x) = x^2$, $\alpha = 0$, $\kappa = \sqrt{2}$, $\sigma = 1$, $x_0 = 0$, $a_0 = -5$, $N = 100$, $M = 8$. The probability masses of the recursion-based transition densities are 0.999 and 1.011, respectively, which is an indicator of the accuracy of the recursive algorithm. Circles denote exact results and crosses denote recursion results.

FIGURE 4 Transition densities before and after the Laplace transform for annuity certain.



(a) Comparison between the recursion results of the Laplace transform $\rho^s(x, x_0)$ and the analytically exact calculations given $V(x) = \mathrm{e}^{-x}$, $s = 0.1$, $x_0 = 0$, $a_0 = -5$, $N = 100$. Solid line: exact. Crosses: recursion. (b) Comparison between kernel $k_{1,2}(x, t \mid x_0)$ obtained by Laplace inversion of the recursion and the exact results with exponential potential $V(x) = \mathrm{e}^{-x}$, given $x_0 = 0$, $t = 1$, $a_0 = -5$, $N = 100$, $M = 8$. Circles: exact. Crosses: recursion.

We take $\delta = 0$ for simplicity. Accordingly, the double Laplace transform of the probability density is actually the fundamental solution of the differential equation (De Schepper *et al* (1992)):

$$\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - (s + u\mathrm{e}^{-x})\Psi = 0 \tag{4.3}$$

After introducing the substitution $t = s\sqrt{2u}\mathrm{e}^{-x/2}$, the fundamental solution of (4.3), namely Green's function, can be written as:

$$\Psi(x \mid s,u) = \begin{cases} 4I_{2\sqrt{2s}}(2\sqrt{2u})K_{2\sqrt{2s}}(2\sqrt{2u}\mathrm{e}^{-x/2}), & x < 0 \\ 4K_{2\sqrt{2s}}(2\sqrt{2u})I_{2\sqrt{2s}}(2\sqrt{2u}\mathrm{e}^{-x/2}), & x > 0 \end{cases} \tag{4.4}$$

where, in this example, we take $u = 1$. Upon comparing (4.3) and (2.10) we find that, in this example, effectively, $V(x) = \mathrm{e}^{-x}$. Therefore, we investigate the performance of the recursive algorithm when the potential is governed by an exponential function.

In the following, we first calculate the recursion results using the exponentially distributed potential and compare these results with the analytically exact result (4.4), which is plotted in part (a) of Figure 4. Afterward, we implement the real inverse Laplace transform for both expressions. The results in part (b) of Figure 4 show remarkable agreement.

4.3 Diffusion processes on the positive real line

In this section, we extend the application of the recursive algorithm to diffusion processes that are only supported on the positive real line. We restrict attention to transition densities for unit diffusion processes, since any diffusion process can be transformed into a process with unit diffusion by a change of variables, as we discussed in Section 2. Integration from $-\infty$ to ∞ shows that on the whole real line this is indeed a transition density. We worked out the results numerically and the accuracy may be checked by comparing the total probability mass with unity. For the diffusion process restricted to the positive real line, we essentially face a first-passage-time problem, that is, the total probability mass is actually the probability of a stochastic process reaching end point y at time t without hitting the absorbing barrier at the origin y_0, and this probability mass is not equal to one.

4.3.1 The whole real line

Consider:

$$\mathrm{d}Y_t = A(Y_t)\,\mathrm{d}t + \mathrm{d}W_t, \quad -\infty < Y_t < \infty$$

in which case:

$$p_Y(y,t \mid y_0) = \exp\left\{\int^{y} A(y')\,\mathrm{d}y' - \int^{y_0} A(y')\,\mathrm{d}y'\right\} \times \mathbb{E}^{y,y_0,t}\left[\exp\left\{-\frac{1}{2}\int_0^t \left(A^2(Y_\tau) + \frac{\partial A(Y_\tau)}{\partial Y_\tau}\right)\mathrm{d}\tau\right\}\right]$$

As a specific example, we take $A(Y_t) = Y_t$ so that closed-form expressions can be obtained. Then we can rewrite the above equation as:

$$\begin{aligned} p_Y(y,t \mid y_0) &= \exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\mathbb{E}^{y,y_0,t}\left[\exp\left\{-\frac{1}{2}\int_0^t Y_\tau^2\,\mathrm{d}\tau\right\}\right] \\ &= \exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\frac{1}{\sqrt{2\pi\,\sinh(t)}} \\ &\quad \times \exp\left\{-\frac{1}{2\sinh(t)}[(y^2 + y_0^2)\cosh(t) - 2yy_0]\right\} \\ &= \frac{1}{\sqrt{2\pi\,\sinh(t)}}\mathrm{e}^{-t/2}\exp\left\{-\frac{1}{2\sinh(t)}(y\mathrm{e}^{-t/2} + y_0\mathrm{e}^{t/2})^2\right\} \end{aligned}$$

where the second equality is derived from the Feynman path integral of the harmonic oscillator in quantum mechanics with Lagrangian $L(y) = \frac{1}{2}(\dot{y}^2 + y^2)$ (Feynman and Hibbs (1965)).

4.3.2 The positive real line

Consider $\mathrm{d}Y_t = A(Y_t)\,\mathrm{d}t + \mathrm{d}W_t$, $Y_t > 0$, where we take $A(Y_t) = Y_t$. In this case:

$$
\begin{aligned}
&p_Y(y,t \mid y_0)\\
&= \exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\mathbb{E}^{y,y_0,t}\left[\exp\left\{-\frac{1}{2}\int_0^t Y_\tau^2\,\mathrm{d}\tau\right\} \;\middle|\; Y_\tau > 0\right]\\
&= \lim_{g\to 0}\exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\mathbb{E}^{y,y_0,t}\left[\exp\left\{-\frac{1}{2}\int_0^t Y_\tau^2\,\mathrm{d}\tau - g\int_0^t \frac{\mathrm{d}\tau}{Y_\tau^2}\right\} \;\middle|\; Y_\tau > 0\right]\\
&= \exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\frac{1}{\eta(t)\xi(0) - \eta(0)\xi(t)}\exp\left\{-\frac{1}{2}\frac{\xi(t)\eta'(0) - \eta(t)\xi'(0)}{\eta(t)\xi(0) - \eta(0)\xi(t)}y_0^2\right\}\\
&\qquad\times\exp\left\{-\frac{1}{2}\frac{\xi(0)\eta'(t) - \eta(0)\xi'(t)}{\eta(t)\xi(0) - \eta(0)\xi(t)}y^2\right\}\sqrt{yy_0}I_{1/2}\left(\frac{yy_0}{\eta(t)\xi(0) - \eta(0)\xi(t)}\right)
\end{aligned}
$$

where the last expression is proved in Vanneste *et al* (1994), and $I_\upsilon(z)$ is the modified Bessel function of the first kind of order υ. Moreover, $\eta(\tau)$ and $\xi(\tau)$ are the independent solutions of $\upsilon''(\tau) = \upsilon(\tau)$ with the boundary conditions $\upsilon(0) = 0$, $\upsilon'(0) = 1$ and $\upsilon(0) = 1$, $\upsilon'(0) = 0$. Hence, $\eta(\tau) = \sinh(\tau)$, $\xi(\tau) = \cosh(\tau)$ (see also Borodin and Salminen (2002) and Linetsky (2004c) for related results). After some algebra, the transition densities for the diffusion process on the positive real line can be expressed as:

$$
\begin{aligned}
&p_Y(y,t \mid y_0)\\
&= \exp\{\tfrac{1}{2}y^2 - \tfrac{1}{2}y_0^2 - \tfrac{1}{2}t\}\frac{1}{\sinh(t)}\exp\left\{-\frac{1}{2}\frac{\cosh(t)}{\sinh(t)}(y_0^2 + y^2)\right\}\sqrt{yy_0}I_{1/2}\left(\frac{yy_0}{\sinh(t)}\right)\\
&= \frac{\mathrm{e}^{-t/2}}{\sqrt{2\pi\sinh(t)}}\left[\exp\left\{-\frac{(y\mathrm{e}^{-t/2} - y_0\mathrm{e}^{t/2})^2}{2\sinh(t)}\right\} - \exp\left\{-\frac{(y\mathrm{e}^{-t/2} + y_t\mathrm{e}^{t/2})^2}{2\sinh(t)}\right\}\right]
\end{aligned}
$$

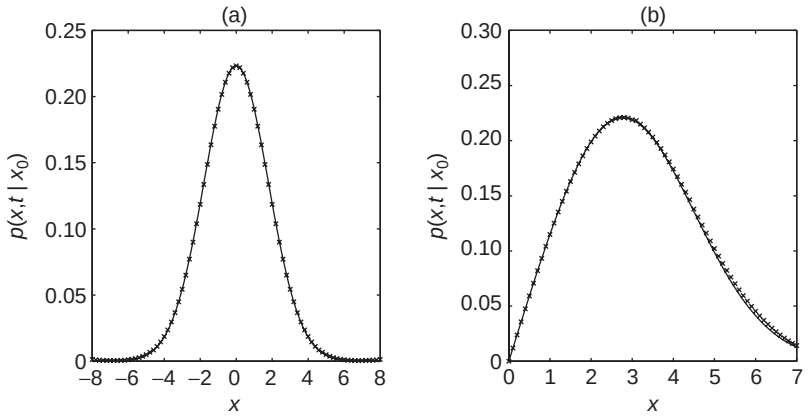
Numerical results comparing the recursive scheme and the closed-form results are plotted in Figure 5 on the facing page.

4.4 The double-well potential model

As an example that does not permit a closed-form solution, we may consider the double-well potential model:

$$\mathrm{d}X_t = (X_t - X_t^3)\,\mathrm{d}t + \mathrm{d}W_t$$

FIGURE 5 Transition densities for diffusions on the whole real line and the positive real line.



(a) Transition probability density for the unit diffusion process on the whole real line given $x_0 = 0$, $t = 1$, $a_0 = -10$, $N = 100$, $M = 8$. The total probability mass of the recursive scheme is 1.001, which, compared with 1, measures the accuracy. (b) Transition probability density for the unit diffusion process on the positive real line given $x_0 = 1$, $t = 1$, $a_0 = -15$, $N = 150$, $M = 8$. Since the diffusion process on the positive real line can be regarded as a first-passage problem, the corresponding probability mass will not be equal to one. Solid line: exact. Crosses: recursion.

with potential $V(x) = \frac{1}{2}(x^6 - 2x^4 - 2x^2 + 1)$. According to Remark 3.2, we can express the transition probability of this model as:

$$p_X(x,t \mid x_0) = \exp\{\tfrac{1}{2}(x^2 - x_0^2) - \tfrac{1}{4}(x^4 - x_0^4) - tV_{\min}\} \times \mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t (V(X_\tau) - V_{\min})\,\mathrm{d}\tau\right\}\right]$$

where $V_{\min}$ is the global minimum of the potential. Figure 6 on the next page reports for $t = 1$ the comparison between the results obtained by the recursive algorithm and by an approximation based on Jensen's inequality, which is discussed in detail by Goovaerts *et al* (2004).

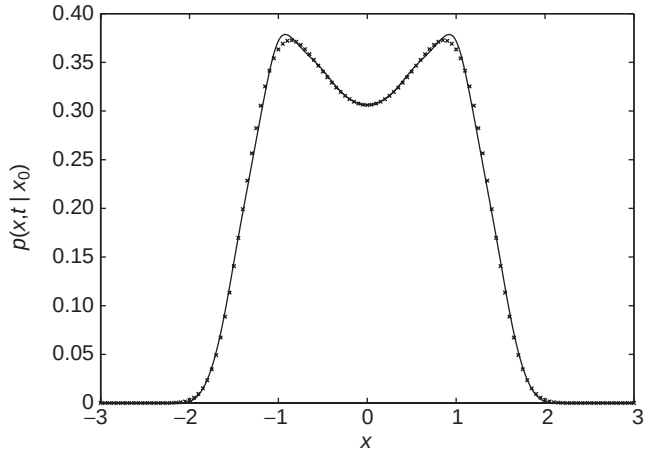
4.5 The CKLS model

The dynamics of the CKLS interest rate model (Chan *et al* (1992)) are formulated as:

$$\mathrm{d}X_t = \kappa(\alpha - X_t)\,\mathrm{d}t + \sigma X_t^{\rho}\,\mathrm{d}W_t \tag{4.5}$$

This model can be regarded as a generalization of some popular interest rate models, for example, Vasicek, Cox–Ingersoll–Ross (CIR) and constant elasticity of variance.

FIGURE 6 Transition densities under the double-well potential model.



Transition probability density under the double-well potential model given $x_0 = 0$, $t = 1$, $a_0 = -5$, $N = 200$, $M = 8$. The total probability mass for the recursive scheme equals 1.000, which is an indicator of the accuracy of the recursive algorithm. Solid line: lower bound approximation. Crosses: recursion. The lower bound approximations are based on an application of Jensen's inequality. Contrary to what the designation "lower bound" suggests, due to scaling, this approximation is not a true lower bound to the transition density.

In this example, similar to the unit diffusion process on the positive real line, the domain is restricted to the positive real line only and we will use the following values for the parameters, as in Aït-Sahalia (1999):

$$\kappa = 0.0972, \qquad \alpha = 0.0808, \qquad \sigma = 0.7224, \qquad \rho = 1.5$$

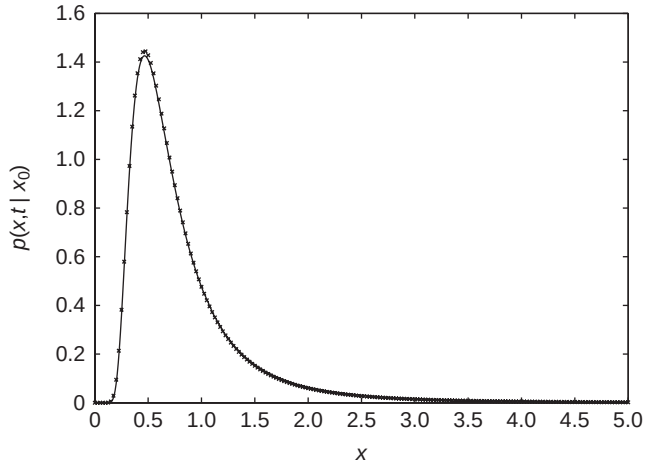
The CKLS model is another example that does not allow an explicit closed-form solution.

The Feynman–Kac integral representation of the transition density is:

$$\begin{aligned} &p_X(x,t \mid x_0) \\ &\quad = \frac{\mathrm{e}^{-\kappa t}}{\sigma x^{3/2}}\left(\frac{x_0}{x}\right)^{3/4} \exp\left\{\frac{\kappa}{\sigma^2}\left(\frac{1}{x}-\frac{1}{x_0}\right) - \frac{\kappa\alpha}{2\sigma^2}\left(\frac{1}{x^2}-\frac{1}{x_0^2}\right)\right\} \\ &\qquad \times \mathbb{E}^{-2/\sigma\sqrt{x},-2/\sigma\sqrt{x_0},t}\left[\exp\left\{-\frac{1}{8}\int_0^t \left[\frac{3}{X_\tau^2} + (\kappa^2 - 3\kappa\alpha\sigma^2)X_\tau^2 - \tfrac{1}{2}\kappa^2\alpha\sigma^2 X_\tau^4 \right.\right.\right. \\ &\qquad\qquad \left.\left.\left. + \tfrac{1}{16}\kappa^2\alpha^2\sigma^4 X_\tau^6\right]\mathrm{d}\tau\right\} \;\middle|\; X_\tau < 0\right] \end{aligned}$$

Figure 7 on the facing page compares the results obtained by the recursive algorithm for $t = 1$ with those obtained by the approximation in Goovaerts *et al* (2004).

FIGURE 7 Transition densities under the CKLS model.



Transition probability density under the CKLS model given $x_0 = 1$, $t = 1$, $a_0 = -5$, $N = 400$, $M = 8$. The total probability mass for the recursive scheme equals 1.003, which is an indicator of the accuracy of the recursive algorithm. Solid line: lower bound approximation. Crosses: recursion. The lower bound approximations are based on an application of Jensen's inequality. Contrary to what the designation "lower bound" suggests, due to scaling, this approximation is not a true lower bound to the transition density.

Next, we consider the pricing of a caplet on a three-month LIBOR rate, where the three-month LIBOR rate r_t is modeled by the short rate of the CKLS model:

$$\mathrm{d}r_t = \kappa(\alpha - r_t)\,\mathrm{d}t + \sigma r_t^{\rho}\,\mathrm{d}W_t$$

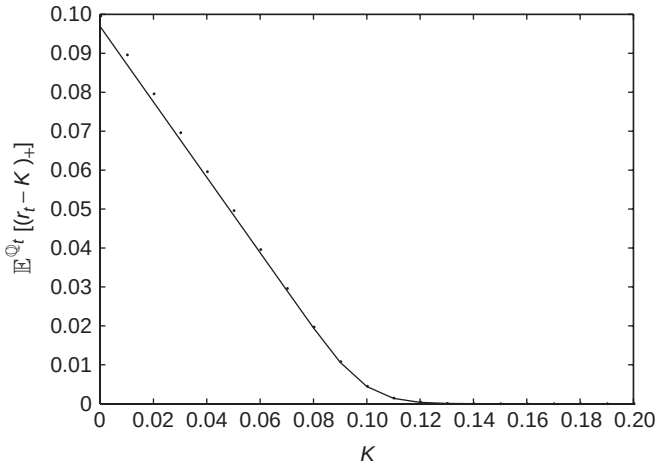
As is common, we work in a world that is always forward risk neutral with respect to a bond maturing at the next reset date. The value of the caplet is then given by:

$$P(0,t)L\Delta\mathbb{E}^{\mathbb{Q}_t}[(r_t - K)_+] = P(0,t)L\Delta\int_0^{\infty} p^{\mathbb{Q}_t}(r,t \mid r_0)(r-K)_+\,\mathrm{d}r$$

where $P(0,t)$ is the price at time 0 of a zero-coupon bond paying off 1 unit at time t, $\mathbb{E}^{\mathbb{Q}_t}$ is the forward risk neutral expectations operator, and $L\Delta(r_t - K)_+$ is the payoff of the derivative product at time t, where L is the principal amount, Δ is the time between reset dates, and K is the cap rate. (For simplicity, we ignore the typical gap between the time at which an interest rate is observed and the time at which the payoff is due.) We take the following parameters, similarly to above:

$$\kappa = 0.0972, \qquad \alpha = 0.0808, \qquad \sigma = 0.7224, \qquad \rho = 1.5$$
$$r_0 = 0.1, \qquad t = 0.25, \qquad a_0 = -0.3, \qquad N = 200, \qquad M = 8$$

FIGURE 8 Caplet pricing under the CKLS model.



Caplet pricing under the CKLS model given $\kappa = 0.0972$, $\alpha = 0.0808$, $\sigma = 0.7224$, $\rho = 1.5$, $r_0 = 0.1$, $t = 0.25$, $a_0 = -0.3$, $N = 200$, $M = 8$. The figure plots the expectation $\mathbb{E}^{\mathbb{Q}_t}[(r_t - K)_+]$ as a function of K, obtained from recursion and Monte Carlo simulation. Solid line: recursion. Dots: simulation.

The recursion results for the expectation $\mathbb{E}^{\mathbb{Q}_t}[(r_t - K)_+]$ and the comparison with Monte Carlo simulation results are plotted in Figure 8, as a function of K. We find that the recursive scheme again produces highly accurate results. In terms of computational efficiency, we note that the recursive scheme with Laplace inversion already directly generates the required ingredients for the numerical approximation of the integral by a sum, which therefore constitutes only a mild computational burden.

5 EXTENSION TO FUNCTIONALS OF NON-GAUSSIAN PROCESSES

One of the main imperfections undermining the Black–Scholes–Merton model is that asset log returns typically do not seem to follow a Gaussian distribution, as reflected in their asymmetry and high kurtosis, and are conceivably not even generated from a continuous diffusion. In this section, we extend our recursive scheme to include functionals of some non-Gaussian processes.

Consider a symmetric stable Lévy process X_t with characteristic function $\varphi_{X_t}(\eta)$ given by:

$$\begin{aligned}\varphi_{X_t}(\eta) &= \mathbb{E}[\exp\{\mathrm{i}\eta X_t\}] \\ &= \exp\{-\tfrac{1}{2}t|\eta|^{\alpha}\}\end{aligned}$$

where α, $0 < \alpha \leqslant 2$, is the index of stability (see, for example, Samorodnitsky and Taqqu (1994) for further details). The probability density function $f(x;\alpha,t)$ of the symmetric stable Lévy process is obtained by the Fourier formula:

$$f(x;\alpha,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{X_t}(\eta)\mathrm{e}^{-\mathrm{i}x\eta}\,\mathrm{d}\eta$$

An important property of a Lévy process in law is that it is infinitely divisible (see, for example, Sato (1999) for further details).

Hence, for some piecewise continuous and nonnegative V, we write:

$$k(x,t \mid x_0) := \mathbb{E}^{x,x_0,t}\left[\exp\left\{-\int_0^t V(X_\tau)\,\mathrm{d}\tau\right\}\right]$$

We adopt again the integral representation (3.3). Considering momentarily a single weighted delta function potential V:

$$\begin{aligned} k(x,t \mid x_0) = \sum_{n=0}^{\infty}(-1)^n \int_0^t \mathrm{d}\tau_n \int_0^{\tau_n} \mathrm{d}\tau_{n-1}\cdots\int_0^{\tau_2}\mathrm{d}\tau_1 \\ \times \int_{-\infty}^{\infty}\mathrm{d}x_n\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\; f(x,x_n;t-\tau_n)V(x_n)\cdots V(x_1)f(x_1,x_0;\tau_1) \end{aligned}$$

We take the Laplace transform of the transition probability with respect to time t, writing:

$$\rho_1^s(x_i,x_j) := \int_0^{\infty}\mathrm{e}^{-st}f(x_i,x_j;t)\,\mathrm{d}t$$

and analogously for k:

$$\rho^s(x,x_0) := \int_0^{\infty}\mathrm{e}^{-st}k(x,t \mid x_0)\,\mathrm{d}t$$

similar to the case of time-homogeneous diffusion processes. We then derive a recursive scheme for the computation of ρ^s. The availability of a closed-form expression for $\rho_1^s(x_i,x_j)$ will provide the initial value of the recursion. Specifically:

$$\rho^s(x,x_0) = \sum_{n=0}^{\infty}(-1)^n\int_{-\infty}^{\infty}\mathrm{d}x_n\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\,\rho_1^s(x,x_n)V(x_n)\cdots V(x_1)\rho_1^s(x_1,x_0) \tag{5.1}$$

Therefore, for an arbitrary piecewise continuous and nonnegative V, we obtain a similar recursive scheme:

$$\begin{aligned} &\rho^{s,(m+1)}(x,x_0) \\ &\quad = \rho^{s,(m)}(x,x_0) - \frac{\rho^{s,(m)}(x,a_{m+1})\rho^{s,(m)}(a_{m+1},x_0)V(a_{m+1})(a_{m+2}-a_{m+1})}{1+\rho^{s,(m)}(a_{m+1},a_{m+1})V(a_{m+1})(a_{m+2}-a_{m+1})} \end{aligned}$$

with initial value:

$$\rho^{s,(0)}(x,x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(x-x_0)\eta}}{s+\frac{1}{2}|\eta|^{\alpha}}\,\mathrm{d}\eta$$

Kac (1951) verified one possible explicit result for the distribution of the functional $\int_0^t V(X_\tau)\,\mathrm{d}\tau$ by transforming the problem into solving the corresponding differential or integral equations. Recall that, by the generalized central limit theorem, stable distributions are the only distributions that can be obtained as limits of normalized sums of independent and identically distributed random variables. Specifically, Kac (1951) considered the case in which the potential V is defined as a Heaviside function, that is:

$$V(x)=\tfrac{1}{2}(1+\operatorname{sgn}(x))=\begin{cases}0, & x<0\\ \frac{1}{2}, & x=0\\ 1, & x>0\end{cases}$$

The "mass" of the Laplace transform of k can then be calculated explicitly and is given by:

$$\int_{-\infty}^{\infty}\rho^{s}(x,x_0)\,\mathrm{d}x=\frac{1}{\sqrt{s(s+1)}}\tag{5.2}$$

After double Laplace inversion, the distribution function of the functional $\int_0^t V(X_\tau)\,\mathrm{d}\tau$ can then be expressed by an arcsine law:

$$\operatorname{prob}\left\{\int_0^t V(X_\tau)\,\mathrm{d}\tau<v\right\}=\frac{2}{\pi}\arcsin\sqrt{\frac{v}{t}},\quad 0\leqslant v\leqslant t$$

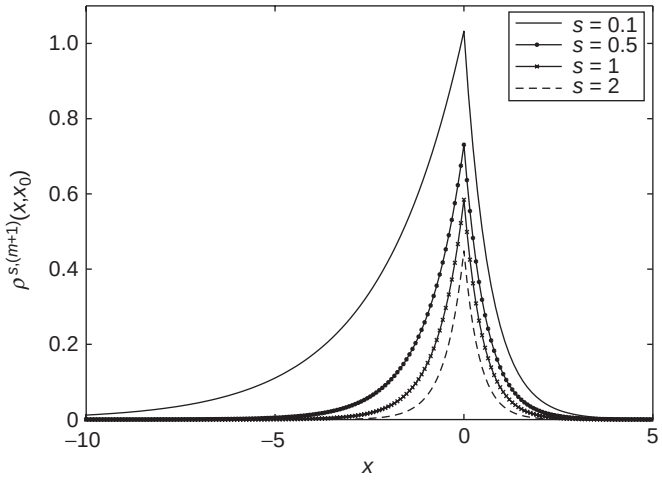
which coincides with the Gaussian case ($\alpha=2$) (see also Linetsky (1999) and Davydov and Linetsky (2002) for financial applications of this distribution of occupation times of Brownian motion), but holds more generally for symmetric stable Lévy processes.

Recursion results for $\rho^s(x,x_0)$ are plotted in Figure 9 on the facing page, assuming a Heaviside potential and taking $\alpha=1.5$ in the initial value of the recursion. In Table 5 on page 76, we list the mass of the recursion-based Laplace transform of k for different choices of parameter values and compare these values with the analytically exact values (5.2). With the increase of N, the recursive scheme converges to the analytically exact value, which shows that the recursive scheme also works properly for functionals of stable Lévy processes.

Finally, we extend our recursion results to the case in which V is given by the generalized Heaviside function:

$$V(x)=\tfrac{1}{2}(1+\operatorname{sgn}(x-a))=\begin{cases}0, & x<a\\ \frac{1}{2}, & x=a\\ 1, & x>a\end{cases}$$

FIGURE 9 The Laplace transformed kernel given the Heaviside potential.



The Laplace transform of k for the Heaviside potential, given $x_0 = 0$, $\alpha = 1.5$, $a_0 = -15$, $N = 250$, $s = 0.1, 0.5, 1, 2$.

In the original formulation of Kac (1951), $a = 0$ is assumed, and the author explicitly mentions that his method fails whenever $a \neq 0$. Figure 10 on page 77 plots the results of our recursive scheme, which is still applicable in this setting, for $\rho^s(x, x_0)$, under different choices of a given $s = 0.1$.

6 CONCLUSION

In most cases, the transition density function of an Ito stochastic differential equation is not available in closed form. Using Feynman–Kac integration, we have constructed a recursive scheme for the Laplace transform (in time) of the transition density function. Various examples illustrate its high degree of accuracy. Generalizations of our technique to functionals of some non-Gaussian processes have also been put forward. Applications of our method to other asset pricing problems and to more general stochastic processes will be investigated further in future research.

Throughout, we have restricted attention to variables in one (spatial) dimension. A crucial role in the derivation of the Feynman–Kac formula (2.8), and hence also in the development of our recursive scheme, is played by the Lamperti transform (2.1). While every one-dimensional diffusion is, in principle, "reducible" to unit diffusion by means of the Lamperti transform, not every multidimensional diffusion is. Generalizations of our recursive scheme to allow for reducible multidimensional diffusions are, in

TABLE 5 Extension to the non-Gaussian process.

s	N	$\lvert a_0\rvert$	Recursion	Exact value
0.1	100	15	2.9794	3.0151
	150	15	2.9974	3.0151
	250	15	3.0069	3.0151
0.5	100	10	1.1541	1.1547
	150	10	1.1544	1.1547
	250	10	1.1546	1.1547
1	100	10	0.7094	0.7071
	150	10	0.7081	0.7071
	250	10	0.7074	0.7071
2	100	10	0.4123	0.4082
	150	10	0.4100	0.4082
	250	10	0.4089	0.4082

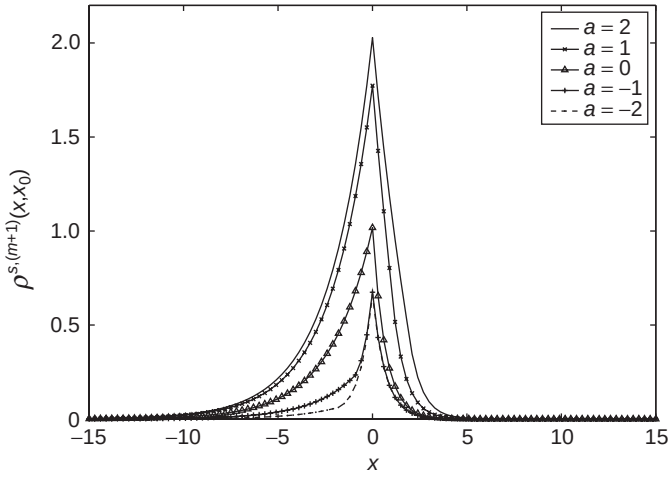
Recursive approximations compared with the analytically exact values obtained from (5.2) for the Heaviside potential, with $s = 0.1, 0.5, 1, 2$ and $N = 100, 150, 250$, given $x_0 = 0, \alpha = 1.5$.

principle, straightforward to obtain. Generalizations to the irreducible case, which will also be applicable to the case in which ψ, and hence T and V, is not explicitly available and that should otherwise be dealt with by numerically approximating V on the partition R, require important changes and would make a good subject for further research.

APPENDIX A. REAL LAPLACE INVERSION USING GAVER FUNCTIONALS

Different numerical algorithms for inverting Laplace transforms are available in the literature. In general, the most commonly adopted methods can be subdivided into four categories in terms of their basic approach: Fourier series expansion, Laguerre function expansion, combination of Gaver functionals and deformation of the Bromwich contour. For a comprehensive review of the available methods, we refer the interested reader to Davies (2002) and Davies and Martin (1979) (see also Davydov and Linetsky (2002)). However, none of these algorithms can be concluded to be the optimal one given different settings and circumstances. Regarding the Laplace inversion for the functions only known on the real axis, Davies and Martin (1979) showed that the methods based on Chebyshev polynomials and Gaver functionals are the most effective methods for a wide range of real-valued functions. Here we use the numerical

FIGURE 10 The Laplace transformed kernel given the Heaviside potential for different a.



The Laplace transform of k for the generalized Heaviside potential, given $x_0 = 0$, $\alpha = 1.5$, $a_0 = -15$, $N = 400$, $s = 0.1$, $a = -2, -1, 0, 1, 2$.

inverse algorithm based on Gaver (1966) functionals with certain nonlinear acceleration sequence transformations, which was discussed in Valko and Abate (2004a,b).

Consider the Laplace transform:

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

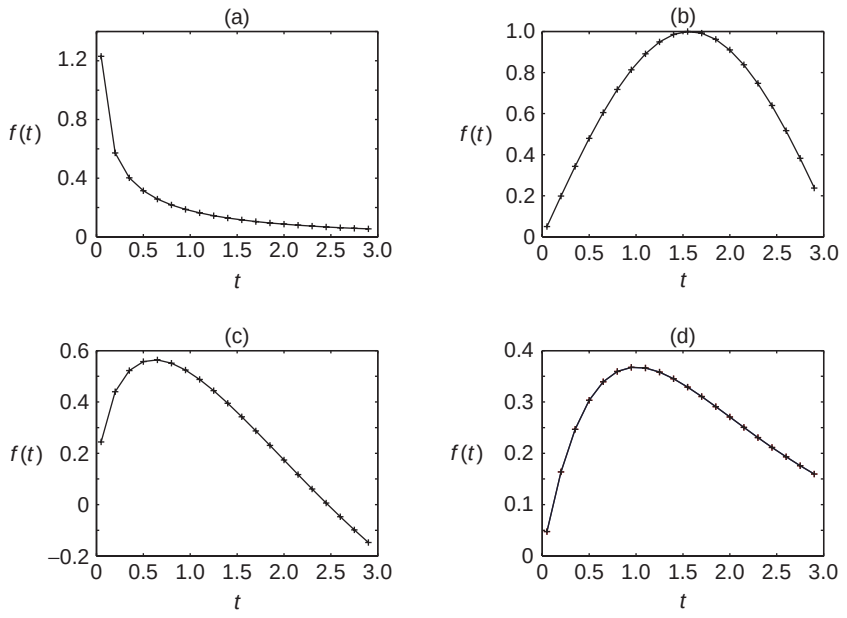
where f is a function of exponential order and, for all $t \geqslant 0$, there exist α, M such that $|F(\tau)| \leqslant M\mathrm{e}^{\alpha\tau}$. The objective of real Laplace inversion is to numerically approximate the real value function f defined on the nonnegative real axis. The analytic solution to the Laplace inversion problem is given by the Post–Widder formula, whose discrete expression is:

$$\begin{aligned} f_k(t) &= (-1)^k k \frac{\ln 2}{t} \binom{2k}{k} \Delta^k F\left(k\frac{\ln 2}{t}\right) \\ &= k\frac{\ln 2}{t} \binom{2k}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} F\left((k+i)\frac{\ln 2}{t}\right) \end{aligned}$$

where Δ is the difference operator, $\Delta F(nx) := F((n+1)x) - F(nx)$.

Theoretically, we may approximate the original function $f(t)$ by a Gaver functional $f_k(t)$ when $k \to \infty$. However, as $k \to \infty$, the lump sum of the Gaver functionals

FIGURE 11 Real Laplace inversion using Gaver functionals based on the functions listed in Table 6 on the facing page for $M = 8$.



The solid lines correspond to the original function $f(t)$ and the crosses correspond to the values of real Laplace inversion approximations of $F(s)$.

converges quite slowly since $|f_k(t) - f(t)| \sim c/k$. Appropriate convergence acceleration algorithms have to be employed. Valko and Abate (2004a) showed that the nonlinear method, Wynn's rho algorithm, is the best among several candidates. Wynn's rho algorithm is based on the Thiele interpolation formula, and is recurrently defined as:

$$\rho_k^{(n)} = \rho_{k-2}^{(n+1)} + \frac{k}{\rho_{k-1}^{(n+1)} - \rho_{k-1}^{(n)}}, \quad k \geqslant 1$$

$$\rho_{-1}^{(n)} = 0, \quad \rho_0^{(n)} = f_n(t), \quad n \geqslant 0$$

The approximation is given by $f(t) = \rho_M^{(0)}$, where the integer M is analogous to the terms of the polynomials used in the approximation and it must be an even number. The precision level will increase with M, but with the compensation of computation costs. However, to a certain level, we have to implement the multiple-precision environment to guarantee the stability of the algorithm due to the limited machine precision. For more details on Wynn's rho algorithm, we refer the reader to Wimp (1981).

TABLE 6 List of functions tested, their Laplace inverses and computation time in seconds.

	$f(t)$	$F(s)$	Computation time (s)
I	$\frac{1-\mathrm{e}^{-t}}{2\sqrt{\pi t^3}}$	$\frac{1}{\sqrt{s}+\sqrt{s+1}}$	0.078
II	$\sin(t)$	$\frac{1}{1+s^2}$	0.094
III	$\frac{\sin 2\sqrt{t}}{\sqrt{\pi}}$	$\frac{\mathrm{e}^{-1/s}}{\sqrt{s^3}}$	0.093
IV	$t\mathrm{e}^{-t}$	$\frac{1}{(1+s)^2}$	0.079

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