

General short-rate analytics

Alexandre Antonov and Michael Spector present an analytical approximation of zero-coupon bonds and swaption prices for general short-rate models. The approximation is based on regular and singular expansions with respect to low volatility and contains a low-dimensional integration. The model in hand assumes the short rate is an arbitrary function of a multi-dimensional Gaussian underlying process. The high approximation accuracy is confirmed by numerical experiments

Short-rate models underlie the first steps of quantitative finance development. The main feature of such models consists of postulating the short-rate process. Vasicek (1977) and Hull & White (1990) pioneered a Gaussian short-rate model still popular among practitioners due to its analytical tractability and transparency. Black & Karasinski (1991) have proposed a log-normal short-rate model. Both models share the same Gaussian mean-reverting process but with different interpretations in terms of the short rate. Later, there appeared multi-factor generalisations (see, for example, Hull & White, 1994) as well as functional generalisations (for example, the generalised Black-Karasinski (BK) (Tourrucôo, Hagan & Schleiniger, 2007) and quadratic short-rate models (Longstaff, 1989, Beaglehole & Tenney, 1992, Jamshidian, 1996 and Piterbarg, 2009)). One can also mention the exactly solvable Cox-Ingersoll-Ross (CIR) short-rate model (1985).

In this article, we consider a multi-factor mean-reverting Gaussian underlying process and view the short rate as being an arbitrary function of it. We refer to such models as generalised short-rate models with Gaussian underlying (GSRGs). Note that the CIR model remains slightly apart from the GSRG framework because its short rate cannot be represented as a deterministic function of a Gaussian process.

In general, the GSRGs do not permit exact analytical interpretation for both zero-coupon bonds and swaption prices¹ but have attractive features of implied volatility skew control. Note that the classic Hull & White model has a fixed normal skew and the BK model possesses the almost-flat implied volatility form due to its quasi-lognormality.

The current post-crisis market favours simple models having several essential elements such as the presence of a multi-factor underlying (to treat correlations) and the implied volatility skew control. The usage of the arbitrage-free models was recently extended to economic scenario generation, especially in the insurance industry. However, the standard models lead to undesirably extreme rates. Thus, a short-rate model with bounded evolution of the rates can be attractive for practitioners.

The GSRGs can satisfy these properties and thus become adequate for the current market environment. What has been lacking is analytical tractability. Indeed, in spite of the simplicity of the lattice implementation for the GSRG, analytical expressions for swaption prices necessary for efficient calibration remain obscure. A step forward in this direction is the analytical

expression for zero-coupon bonds for the generalised BK model. Tourrucôo, Hagan & Schleiniger (2007) have derived the analytical approximation for the one-factor case.

In this article, we go further and come up with the multi-dimensional case analytical formulas for both zero-coupon bond and European swaption prices. Differences from the Tourrucôo, Hagan & Schleiniger approach, such as the underlying process drift adaptation to the yield curve, will be explained.

We also should mention an earlier work of Hansen & Jørgensen, 1998, where the authors approximate the zero bond for one-factor log-normal short rate model with time-independent drift and volatility. The authors present a zero bond in terms of power series in the short rate with multi-dimensional ordered integrals as coefficients. The method is hardly efficiently generalizable for time-dependent drift and volatility which leads to impossibility to fit a yield curve. Moreover, a bad convergence of the series for high values of the short rate makes the approximation application limited.

We illustrate the method with two examples. The first is our version of the generalised multi-factor BK model allowing for skew control. The second is a new bounded short-rate model where the rates evolve between certain user-defined limits.

The article is organised as follows. First, we introduce notations, present swaption pricing logic and derive the main partial differential equations (PDEs). Then we present the main results: quadrature formulas for the zero-coupon bond and the Arrow-Debreu (AD) price. We give two examples of the short-rate models: our version of the generalised multi-factor BK model and the bounded short-rate model. Finally, we present a numerical confirmation of the approximation accuracy, followed by conclusions.

Preliminary remarks

Here, we introduce the main notations, derive the PDEs and determine the expansion strategy.

Denote an underlying F -factor mean-reverting process as:

$$dx_i(t) = (\phi_i(t) - a_i(t)x_i(t))dt + \lambda_i(t) \cdot dW(t) \quad (1)$$

for $i = 1, \dots, F$

for the vector volatilities $\lambda_i(t)$, vector uncorrelated Brownian motions $dW(t)$, mean-reversions $a_i(t)$ and drifts $\phi_i(t)$, serving to link the short-rate process to a given yield curve. The dot operation \cdot defines the vector inner product.

¹ Except the Hull-White model and the Piterbarg quadratic model (Piterbarg, 2009)

An equivalent, alternative form of the underlying can contain correlated scalar Brownian motions $dW_i(t)$ and scalar volatilities $\gamma_i(t)$, that is, $dx_i(t) = (\phi_i(t) - a_i(t)x_i(t))dt + \gamma_i(t)dW_i(t)$.

We postulate the short-rate process as an arbitrary function of the underlying processes:

$$r(t) = f(t, x) \tag{2}$$

Having sufficient freedom, we choose the zero initial values $x_i(0) = 0$. Now we define the numeraire process, a savings account, $N_t = e^{\int_0^t dsr(s)}$. A zero-coupon bond with maturity T equals:

$$P(t, T) = N_t E \left[\frac{1}{N_T} \middle| F_t \right] = E \left[e^{-\int_t^T dsr(s)} \right]$$

In other words, the zero-coupon bond is a function of the underlying process. Here, and throughout this article, we refer to the operator $E[\dots]$ as the expectation in the risk-neutral measure.

The discount factor $D(T) \equiv P(0, T)$ is a zero-coupon bond maturing at T as seen from the origin. Denote today's forward rate as $R(t)$:

$$D(T) = e^{-\int_0^T dsR(s)}$$

Then, without loss of generality, we can represent² the short rate 'moving' around today's forward rate:

$$f(t, 0) = R(t) \tag{3}$$

The discount factor is considered as a model input and the drift parameters $\phi_i(t)$ should be chosen to fit it:

$$D(T) = E \left[e^{-\int_0^T ds f(s, x(s))} \right] \tag{4}$$

Note that the drifts will have the second order in volatilities due to the condition (3).

It is convenient to remove the mean-reverting term from the underlying x 's using an appropriate multiplier:

$$y_i(t) = x_i(t)A_i(t) \text{ for } A_i(t) = e^{\int_0^t ds a_i(s)}$$

We introduce normal processes y_i satisfying the stochastic differential equation (SDE):

$$dy_i(t) = \theta_i(t)dt + \sigma_i(t) \cdot dW(t) \tag{5}$$

where $\theta_i(t) = \phi_i(t)A_i(t)$ and $\sigma_i(t) = \lambda_i(t)A_i(t)$. Obviously, the processes start with zero values, $y_i(0) = 0$.

The short rate (2) in terms of the Gaussian y 's will look as follows:

$$r(t) = r(t, y) \equiv f \left(t, \left\{ A_i(t)^{-1} y_i(t) \right\} \right) \tag{6}$$

To calculate a generalised option price with payout at time T :

$$V = E \left[\frac{\left(\sum_n \pi_n P(T, T_n) \right)^+}{N_T} \right]$$

one should be able to evaluate the expectation:

$$V = E \left[\frac{\left(\sum_n \pi_n P(T, T_n) \right)^+}{N_T} \right]$$

Note that the payment coefficients π_n could have opposite signs for non-trivial optionality.

After some experiments with the measure choices, we have concluded that the most efficient way is to proceed in the risk-neutral measure. We use the AD price $q(t, u)$:

$$q(t, u) \equiv E \left[\delta(y(t) - u) \right] = E \left[e^{-\int_0^t dsr(s)} \middle| y(t) = u \right] E$$

where u is a vector and the multi-dimensional delta function is defined by the product $\delta(y(t) - u) = \prod_{i=1}^n \delta(y_i(t) - u_i)$. It permits us to represent the option price as a multi-dimensional integral:

$$V = \int du \left(\sum_n \pi_n P(T, T_n; u) \right)^+ q(T, u) \tag{7}$$

Thus, our goal is to approximate the zero-coupon bond price $P(t, T; u)$ and the AD price $q(t, u)$ as functions of arbitrary u . Then, the integration (7) will be performed numerically.

At the end of this section, we present differential equations for both components of the interest. The zero-coupon bond price corresponds to a backward PDE, $(\partial_t + L(t, u))P(t, T; u) = 0$, with operator:

$$L(t, u) = -r(t, u) + \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j \tag{8}$$

and the final conditions $P(T, T; u) = 1$. We have denoted space derivatives as $\partial_i \equiv \partial_{u_i}$ and the instantaneous covariance matrix as $C_{ij}(t) = \sigma_i(t) \cdot \sigma_j(t)$. The backward PDE can be easily derived by Itô's formula requiring zero drift of the martingale $e^{-\int_0^t dsr(s)} P(t, T; y(t))$.

The AD price, $q(t, u)$, satisfies the conjugate forward PDE, $(\partial_t - L^*(t, u))q(t, u) = 0$, for the operator:

$$L^*(t, u) = -r(t, u) - \sum_i \theta_i(t) \partial_i + \frac{1}{2} \sum_{i,j} C_{i,j}(t) \partial_i \partial_j \tag{9}$$

with the initial condition $q(0, u) = \prod_i \delta(u_i - y_i(0))$. Indeed, the integral $P(0, T) = \int du P(t, T; u)q(t, u)$ is independent of time t . After its differentiation over time, we obtain the desired forward PDE.

To derive our approximation, we will apply the perturbation technique for small volatilities to the PDEs above. With the yield curve fit (4) the leading order of the drifts θ_i will be σ^2 , because of the invariance $\sigma \rightarrow -\sigma$. Thus, we will consider the transformation $\theta \rightarrow \epsilon\theta$ and $\sigma^2 \rightarrow \epsilon\sigma^2$ in the backward and forward PDEs.

Main results

In this section, we present the main results: quadrature formulas for the zero-coupon bond and the AD price. We use two types of expansion technique: regular for the zero-coupon bonds and singular for the AD price.

■ **Zero-coupon bond.** Introducing ϵ in the backward parabolic

² One can use a proper shift, $x_i(t) \rightarrow x_i(t) + \int_0^t ds \psi_i(s)$, for some functions $\psi_i(s)$

³ More details can be found in Antonov & Spector (2010)

operator:

$$L(t, u) = -r(t, u) + \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j \quad (10)$$

we denote the unperturbed operator as $L_0(t)$ and a perturbation as $L_1(t)$:

$$L_0(t) = -r(t, u) \quad (11)$$

$$L_1(t) = \varepsilon \sum_i \theta_i(t) \partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t) \partial_i \partial_j \quad (12)$$

The solution of the unperturbed equation $(\partial_t + L_0(t))P_0(t, T, u) = 0$ for the unit final value, $P_0(T, T, u) = 1$, is very simple:

$$P_0(t, T, u) = \exp\left(-\int_t^T d\tau r(\tau, u)\right) \quad (13)$$

The first and second corrections can be obtained using the regular perturbation technique³:

$$P_1(t, T, u) = \varepsilon P_0(t, T, u) \times \int_t^T ds \left\{ -\sum_i \theta_i(s) g_i(s, T, u) + \frac{1}{2} \sum_{i,j} C_{ij}(s) (g_i(s, T, u) g_j(s, T, u) - g_{ij}(s, T, u)) \right\} \quad (14)$$

with:

$$\begin{aligned} g_i(s, T, u) &= \int_s^T d\tau \partial_i r(\tau, u) \quad \text{and} \\ g_{ij}(s, T, u) &= \int_s^T d\tau \partial_i \partial_j r(\tau, u) \end{aligned} \quad (15)$$

Let us stress that the short rate $r(t, u)$ and its derivatives are functions of the normal underlyings y ; in order to handle the initial dependence of the underlying mean-reverting process x one should use formula (6).

Note that Tourrucóo, Hagan and Schleiniger (2007) have presented the zero-coupon bond in exponential form:

$$P(t, T; u) = P_0(t, T, u) \exp\left\{ \varepsilon \Phi_1(t, T; u) + \varepsilon^2 \Phi_2(t, T; u) + O(\varepsilon^3) \right\}$$

which can sometimes deliver explosive values, especially for large maturities and volatilities.

To finalise the zero-coupon bond approximation, we should fix the drifts, unknown so far, to reproduce the yield curve or discount factor $D(T)$. Equivalently, we can calculate the averages $E[y_i(t)] = m_i(t) = \int_0^t ds \theta_i(s)$, requiring the fit:

$$D(T) = P_0(0, T, 0) + P_1(0, T, 0)$$

to the leading order in the volatilities. Noticing that:

$$P_0(0, T, 0) = \exp\left(-\int_0^T d\tau r(\tau, 0)\right) = \exp\left(-\int_0^T d\tau R_\tau\right) = P(0, T)$$

we conclude that, in order to fit the yield curve, we should choose the averages $m_i(t)$ such that the second-order correction equals zero, that is, $P_1(0, T, 0) = 0$ or:

$$\begin{aligned} & \int_0^T ds \sum_i \theta_i(s) g_i(s, T, 0) \\ &= \frac{1}{2} \int_0^T ds \sum_{i,j} C_{ij}(s) (g_i(s, T, 0) g_j(s, T, 0) - g_{ij}(s, T, 0)) \end{aligned}$$

Differentiating in T , we obtain:

$$\begin{aligned} & \sum_i \partial_i r(T, 0) m_i(T) \\ &= \sum_{i,j} \partial_i r(T, 0) \int_0^T d\tau \partial_j r(\tau, 0) V_{ij}(\tau) - \frac{1}{2} \sum_{i,j} \partial_i \partial_j r(T, 0) V_{ij}(T) \end{aligned} \quad (16)$$

where $V_{ij}(T)$ is the y 's covariance:

$$E[(y_i(t) - m_i(t))(y_j(t) - m_j(t))] = V_{ij}(t) = \int_0^t ds C_{ij}(s)$$

Either solution of equation (16) for individual averages $m_i(T)$ is suitable for the approximation. Note that Tourrucóo, Hagan & Schleiniger (2007) have matched the yield curve by a rough multiplicative adjustment that does not fix the underlying drifts.

The second-order correction was calculated in Antonov & Spector (2010). Although, according to our experiments, the second-order approximation (precision $O(\varepsilon^2)$) does not deliver a reasonable quality increase but can substantially slow down the procedure.

Note that the Gaussian short rate models where the short rate is a linear function of the underlying can be recovered by the expansion technique. Indeed, using a fact that all higher derivatives of the short rate over the underlying cancel, we can restore the closed form of the Gaussian model zero bond calculating series of the expansion terms.

■ **AD price.** In the case of the AD price, we deal with a singular perturbation technique for the parabolic operator:

$$L^*(t, u) = -r(t, u) - \varepsilon \sum_i \theta_i(t) \partial_i + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j \quad (17)$$

The AD price satisfies the forward PDE $(\partial_t - L^* \Pi(t, u))q(t, u) = 0$ with the initial condition $q(0, u) = \Pi_i \delta(u_i - y_i(0))$.

Applying the Dyson technique would require careful calculation of all terms in ε due to the singularity of the initial value. Instead, we use the Heat-Kernel singular expansion method.

It is easy to check that the Gaussian density:

$$q_0(t, u) = \frac{\exp\left(-\frac{1}{2} \varepsilon^{-1} (u - \varepsilon m(t))^T V^{-1}(t) (u - \varepsilon m(t))\right)}{(2\pi\varepsilon)^{\frac{n}{2}} |V(t)|^{\frac{1}{2}}} \quad (18)$$

where $m(t)$ is the $y(t)$ centre and $V(t)$ is the covariance in vector/matrix notations, satisfies the PDE without the rate term:

$$\partial_t q_0(t, u) = -\varepsilon \sum_i \theta_i(t) \partial_i q_0(t, u) + \frac{1}{2} \varepsilon \sum_{i,j} C_{ij}(t) \partial_i \partial_j q_0(t, u) \quad (19)$$

Now, following the Heat-Kernel expansion logic, we look for the solution as $q(t, u) = q_0(t, u) \Omega(t, u)$, where $\Omega(t, u)$ is a regular function of ε with the unit initial condition $\Omega(0, u) = 1$. After substitution into the initial PDE, we get the solution for the AD price up to the first order in low volatilities⁴:

⁴ More details can also be found in Antonov & Spector (2010)

$$\begin{aligned}
 q(t,u) &= \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}(u-\varepsilon m(t))^T V^{-1}(t)(u-\varepsilon m(t))\right)}{(2\pi\varepsilon)^{\frac{n}{2}}|V(t)|^{\frac{1}{2}}} \\
 &\times \exp\left(-\int_0^t ds r(s, V(s)V^{-1}(t)u)\right) \\
 &\times \left\{1 + \varepsilon \int_0^t d\tau (\theta^T(\tau) - m^T(\tau)V^{-1}(\tau)C(\tau)) \right. \\
 &\quad \left. V^{-1}(\tau)\hat{h}^{(1)}(\tau; t, u) \frac{1}{2}\varepsilon \int_0^t d\tau \tau r V^{-1}(\tau)C(\tau)V^{-1}(\tau) \right. \\
 &\quad \left. \left[\hat{h}^{(1)}(\tau; t, u) \otimes \hat{h}^{(1)}(\tau; t, u) - \hat{h}^{(2)}(\tau; t, u)\right]\right\}
 \end{aligned} \tag{20}$$

where vector:

$$\hat{h}_n^{(1)}(\tau; t, u) = \sum_j \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k r(s, V(s)V^{-1}(t)u) \tag{21}$$

and matrix:

$$\hat{h}_{nl}^{(2)}(\tau; t, u) = \sum_{km} \int_0^\tau ds V_{nk}(s) \tilde{\partial}_k \tilde{\partial}_m r(s, V(s)V^{-1}(t)u) V_{ml}(s) \tag{22}$$

Note that the main formula contains vector/matrix notations including standard linear algebra products. For example, outer product $a \otimes b$ of two vectors means a matrix with elements $a_i b_j$. Here, the derivative $\tilde{\partial}_k$ denotes the short-rate derivative over the space argument:

$$\tilde{\partial}_i r(s, z(u)) = \frac{\partial r(s, z(u))}{\partial z_i(u)}$$

Model examples

Here, we give two GSRG examples. One is our version of the generalised multi-factor BK model and the other is the bounded short-rate model, which can be suitable for scenario generation where the rates evolve inside given boundaries.

We will use a symmetric form of the GSRGs:

$$r(t) = f\left(t, \sum_i x_i\right)$$

As mentioned above, we can always represent the short-rate function $f(t, x)$ as being equal to the forward rate for zero values of the underlying process x . Similar scaling arguments can be applied to set the space derivatives of the short rate to one for $x = 0$. These two properties, $f(t, 0) = R(t)$ and $\partial_x f(t, 0) = 1$, are convenient for analysing the calibration results. Indeed, typical underlying x parameters correspond to the normal case, that is, volatilities have 1–1.5% order of magnitude for any short rate.

■ **Generalised multi-factor BK model.** We propose a version of the generalised multi-factor BK model with many attractive features, including swaption skew control and multiple factors permitting decorrelation of the rates. The short rate is presented in the form:

$$r(t) = f(t, x) = R(t) + \frac{e^{S(t)\sum_i x_i} - 1}{S(t)} \tag{23}$$

where $R(t)$ is today's forward rate and positive function $S(t)$ is a shift

parameter controlling the swaption skew. For a special case of $S(t) = R^{-1}(t)$, the model degenerates to the standard multi-factor BK model, $r(t) = R(t)e^{R^{-1}(t)\sum_i x_i}$, whereas when the shift parameter tends to zero, we obtain the Hull-White model, $r(t) = R(t) + \sum_i x_i$.

Note that typical underlying x parameters correspond to the normal case, that is, volatilities have 1–1.5% order of magnitude for any shift. This is another attractive feature of the proposed model.

The analytical expressions can be calculated using the general formulas for the zero-coupon bond (13) and (14) and the AD price (20).

Explicit formulas for the main derivatives (15) are given by:

$$g_i(s, T, u) = \int_s^T d\tau A_i(\tau)^{-1} e^{S_\tau \sum_i A_i(\tau)^{-1} u}$$

and:

$$g_{ij}(s, T, u) = \int_s^T d\tau A_i(\tau)^{-1} A_j(\tau)^{-1} S_\tau e^{S_\tau \sum_i A_i(\tau)^{-1} u}$$

The conditions for the drifts simplify also, and one can choose a 'symmetric' form:

$$m_i(T) = \sum_j \int_0^T d\tau V_{ij}(\tau) A_j^{-1}(\tau) - \frac{1}{2} S_T \sum_j V_{ij}(T) A_j^{-1}(T)$$

■ **Bounded short-rate model.** Sometimes practitioners, especially from the insurance industry, are interested in a bounded rates evolution for scenario generation. The current short-rate models do not meet that bounded requirement. There exists a probability such that rates can exceed an *a priori*-given (positive) value.

We propose the following bounded short-rate model:

$$r(t) = f(t, x) = \frac{U(t)e^{\beta(t)\sum_i x_i} + L(t)\alpha(t)}{e^{\beta(t)\sum_i x_i} + \alpha(t)} \tag{24}$$

where time-dependent functions $L(t)$ and $U(t)$ are the lower and upper boundaries, respectively. As usual, the forward rate is denoted as $R(t)$. To ensure the attractive properties of $f(t, 0) = R(t)$ and $f'(t, 0) = 1$, we define the functions $\alpha(t)$ and $\beta(t)$ as:

$$\alpha(t) = \frac{U(t) - R(t)}{R(t) - L(t)}$$

and:

$$\beta(t) = \frac{U(t) - L(t)}{(R(t) - L(t))(U(t) - R(t))}$$

We see that the function (24) is a monotone function of $X = \sum_i x_i$ and its inverse has a simple form:

$$X = \ln\left(\frac{U(t) - R(t)}{U(t) - F} \frac{F - L(t)}{R(t) - L(t)}\right) \beta^{-1}(t)$$

for $F = f(t, X)$. Obviously, the short rate has two desired limits: $\lim_{X \rightarrow -\infty} f(t, X) = L(t)$ and $\lim_{X \rightarrow +\infty} f(t, X) = U(t)$.

A special case of the bounded short-rate model $U(t) \rightarrow +\infty$ represents the generalised multi-factor BK model (23) for the skew

parameter $S(t) = (R(t) - L(t))^{-1}$. We see that, for this special case, the skewness of the implied volatility depends on the bounds. This property holds also in the general bounded model set-up. To verify it explicitly, consider the short-rate SDE:

$$dr(t) = \dots dt + \frac{(r(t) - L(t))(U(t) - r(t))}{(R(t) - L(t))(U(t) - R(t))} \sum_i dx_i(t)$$

A. Generalised two-factor BK input		
Rate	R(t)	5.00%
Correlation	r(t)	-90.00%
Shift	S(t)	10
Volatility 1	g ₁ (t)	1.00%
Volatility 2	g ₂ (t)	1.75%
Reversion 1	a ₁ (t)	50.00%
Reversion 2	a ₂ (t)	5.00%

B. Generalised two-factor BK results					
Exercise (years)	Length (years)	Strike (%)	Implied volatility (%)		Error (bp)
			Approx	Exact	
5	10	3.28	25.77	25.71	5
5	10	3.67	25.04	24.99	6
5	10	4.10	24.34	24.31	3
5	10	4.59	23.69	23.67	2
5	10	5.13	23.09	23.09	0
5	10	5.73	22.53	22.54	-1
5	10	6.41	22.01	22.01	0
5	10	7.17	21.52	21.50	2
5	10	8.02	21.10	21.06	4
10	10	2.72	24.39	24.26	13
10	10	3.19	23.36	23.27	9
10	10	3.74	22.42	22.34	8
10	10	4.38	21.54	21.47	7
10	10	5.13	20.75	20.67	8
10	10	6.01	20.03	19.96	7
10	10	7.03	19.39	19.32	7
10	10	8.24	18.82	18.75	6
10	10	9.65	18.30	18.25	5
20	10	2.10	21.59	21.24	35
20	10	2.62	20.20	19.89	31
20	10	3.28	18.94	18.68	26
20	10	4.10	17.87	17.62	26
20	10	5.13	16.91	16.68	23
20	10	6.41	16.05	15.86	19
20	10	8.02	15.35	15.15	20
20	10	10.03	14.72	14.57	15
20	10	12.54	14.19	14.09	11

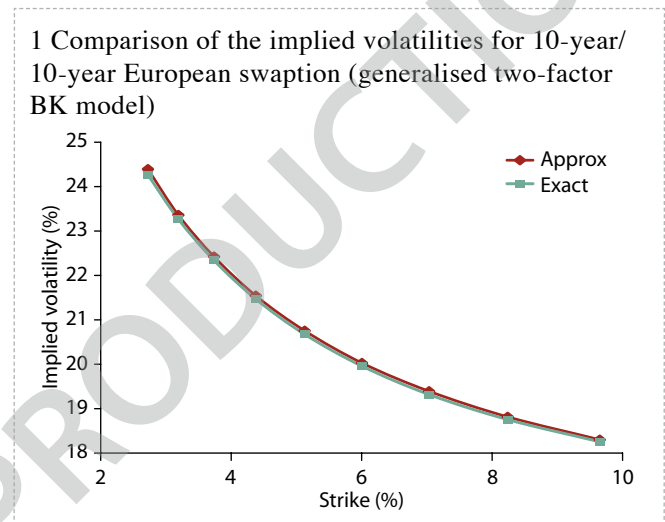
The diffusion coefficient:

$$D(t,r) \sim \frac{(r - L(t))(U(t) - r)}{(R(t) - L(t))(U(t) - R(t))} \sum_i dx_i(t)$$

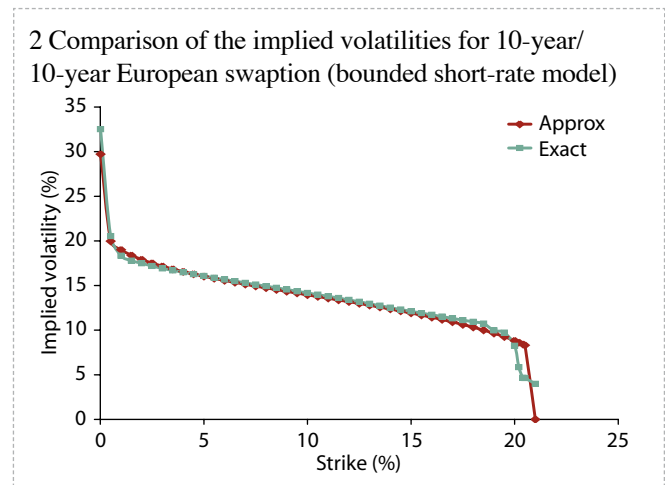
or local volatility, can give an approximate form of the implied volatility smile as a function of strike K:

$$\sigma_I(K) \sim \frac{D\left(t, \frac{R(t)+K}{2}\right)}{\frac{R(t)+K}{2}}$$

We see that the implied volatility form depends on the boundaries. Thus, if one fixes the boundaries, one loses the skew control and the model can be calibrated only to one point of the implied volatility for a given swaption (say at-the-money). On the other hand, fix-



C. Bounded one-factor short-rate model input		
Rate	R(t)	5.00%
Volatility	g(t)	1.5%
Reversion	a(t)	5.00%
Lower bound	L(t)	0%
Upper bound	U(t)	20.00%



ing, say, the lower boundary, one can still calibrate to the option skew using a residual freedom in the upper boundary.

Numerical experiments

Here, we present numerical results for swaption prices comparing our analytical approximation ('Approx') with the exact answer ('Exact') obtained in a high-quality least-squares Monte Carlo simulation.

The approximation is performed in the first order in variances:

$$P(t, T; u) = P_0(t, T; u) + P_1(t, T; u)$$

D. Bounded one-factor short-rate model results			
Strike (%)	Implied volatility (%)		Error (bp)
	Approx	Exact	
0.01	29.72	32.52	-280
0.5	19.98	20.54	-56
1	19.01	18.34	67
1.5	18.39	17.79	60
2	17.92	17.49	43
2.5	17.52	17.19	33
3	17.17	16.92	25
3.5	16.85	16.69	16
4	16.56	16.48	8
4.5	16.29	16.27	2
5	16.04	16.07	-3
5.5	15.8	15.87	-7
6	15.57	15.67	-10
6.5	15.35	15.49	-14
7	15.14	15.3	-16
7.5	14.93	15.11	-18
8	14.74	14.93	-19
9	14.34	14.55	-21
10	13.96	14.17	-21
11	13.58	13.79	-21
12	13.19	13.39	-20
13	12.79	12.96	-17
14	12.37	12.52	-15
15	11.92	12.11	-19
16	11.44	11.72	-28
17	10.92	11.32	-40
18	10.33	10.93	-60
19	9.654	9.972	-31.8
20	8.824	8.254	57
20.2	8.632	5.852	278
20.4	8.425	4.662	376.3
20.5	8.315	4.63	368.5
21	0	3.99	-399

from (13) and (14) for the zero-coupon bond and $q(t, u)$ AD price defined in (20). Note that, according to our experiments, the second-order approximation does not deliver a reasonable quality increase but can substantially slow down the procedure. The integration:

$$V = \int du \left(\sum_n \pi_n P(T, T_n; u) \right)^+ q(T, u) \quad (25)$$

is taken numerically using, for example, an adaptive grid.

Note that the forwards $\int du P(T, T_n; u) q(T, u)$ are not consistent with the theoretical ones, the discount factors $D(T_n)$, due to approximation errors. Thus it can be useful to perform a multiplicative adjustment for the underlying products, $P(T, T_n; u) q(T, u)$, to match the forwards exactly.

■ **Performance.** The numerical integration (25) includes calculation of integrals over time in the zero-coupon bond and the AD price for any fixed space point u . Consider, for example, the zero-bonds. The unperturbed zero-coupon bond integral (13) can be efficiently taken using the Simpson method or using functional proxies for few integration points. The first-order correction (14) accuracy is less important for the convergence due to its small value. Moreover, the underlying two-dimensional integrals contain ordered arguments that lead to linear integration algorithms in the number of time-discretisation points. Note also that certain values, such as the short rate, its derivatives and their integrals, can be pre-calculated from the smallest exercise date until the largest zero-coupon bond maturity and reused in the underlying expressions (13), (14) and (15) for all exercise dates and all zero-coupon bond maturities.

The same arguments apply to the AD price integrals.

The option price approximation (25) containing effectively a space-time integration is definitely more attractive than other numerical methods, including lattices and Monte Carlo simulations especially, for multi-dimensional problems. Depending on the numerical method efficiency, the speed gain can be two to four times for one-dimensional short-rate models and five to 10 times for multi-dimensional cases.

■ **Generalised multi-factor BK model.** We consider a two-factor case of the generalised multi-factor BK model (23):

$$r(t) = R(t) + \frac{e^{S(t)(x_1+x_2)} - 1}{S(t)}$$

with the underlying processes:

$$dx_i(t) = (\phi_i(t) - a_i(t)x_i(t))dt + \gamma_i(t)dW_i(t)$$

for $i = 1, 2$ and $x_i(0) = 0$

with a certain correlation between Brownian motions, $\rho(t) = E[dW_1(t)dW_2(t)]$.

We use standard data, summarised in table A.

We have chosen particular time-independent parameters for transparency. However, the proposed approximation works for general time-independent parameters. Note that the shift parameter corresponds to a skew situated between normal and lognormal ones. Recall that the normal model is described with the zero shift parameter and the lognormal model with the shift equal to the inverse rate, $S(t) = R^{-1}(t) = 20$.

We have tested European swaptions with an annual period, 10-year length, multiple strikes and maturities of five, 10 and 20 years. The results for Black implied volatilities are shown in table B and figure 1. The at-the-money swaptions are marked in bold.

■ **Bounded short-rate model.** We consider a one-factor case of the bounded short-rate model (24):

$$r(t) = f(t, x) = \frac{U(t)e^{\beta(t)x} + L(t)\alpha(t)}{e^{\beta(t)x} + \alpha(t)}$$

with the underlying process:

$$dx(t) = (\phi(t) - a(t)x(t))dt + \gamma(t)dW(t)$$

starting from zero, $x(0) = 0$.

We use standard data, summarised in table C.

We have chosen particular time-independent parameters for transparency. However, the proposed approximation works for general time-independent ones.

We have tested European swaptions with an annual period, 10-year length, multiple strikes and 10-year maturity. The results for Black implied volatilities are presented in figure 2 and table D. One can observe zero implied volatilities for a strike of 21%. This property is also valid for higher strikes due to bounds of rates evolution. Namely, interest rates are highly correlated with the short rate (bounded by 20% in our set-up). Thus we expect the option underlying constant maturity swap 10-year rate to be bounded by a maximum of around 20% of value (around 21% according to the numerical results). This leads to a zero value of a call option for strikes higher than the underlying rate bound.

Conclusion

We have analysed GSRGs and derived an efficient swaption price approximation. The result is based on a regular and singular expansion with respect to low volatility and contains a low-dimensional integration.

We also presented two special cases of the GSRG model. The first is the generalised multi-factor BK model widely used by. The second, the bounded short-rate model, is new. Its rates evolve inside arbitrary thresholds. This model is especially attractive for scenario generation. The implied volatility skew for the bounded short-rate model is

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convex for small/moderate strikes and concave for bigger ones.

We have confirmed the analytical approximation quality by comparison with purely numerical results. We have detected an excellent analytics-numerics fit for short/medium maturities, and a good one for large maturities. We have observed a slight quality degeneration for extreme strikes. ■

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