An analytical framework for credit portfolio risk measures

Monte Carlo simulation of credit-risky portfolios can be computationally intensive when calculating risk measures. Here, ***Mikhail Voropaev*** *builds an analytical framework for calculating value-at-risk and expected shortfall for these portfolios that significantly reduces the required computation*

There is increasing demand for fast and consistent economic capital calculation and allocation techniques. Using industry standard Monte Carlo simulations for portfolio-level risk quantification requires a considerable amount of time and computer power. For risk concentration identification, risk-adjusted pricing and portfolio optimisation, portfolio-wide economic capital needs to be allocated down to individual transactions. This is even more challenging from both the methodological and computational points of view. Statistical noise, an inherent part of Monte Carlo simulations, leads to unstable estimations of the allocated risk (especially in the case of value-at-risk-based capital allocation). Reliable estimations of capital charges based on simulations require significantly more computer time compared with the portfolio-wide calculations.

Techniques such as importance sampling have been developed to improve the performance of the simulations-based approach (see, for example, Kalkbrener, Lotter & Overbeck, 2004). Yet the simulation-based estimation of risk contributions at transaction level is still a demanding computational problem. In practice, the efficiency of these techniques may be limited since it depends on an analytical approximation used to determine a sampling region. Yet another drawback of the simulation-based approach is its inability to efficiently risk-assess new deals in the context of the portfolio.

Although Merton-type models are not analytically tractable in the general case, some progress has been made to develop an approximate solution. The most successful attempts to tackle the problem are the asymptotic single risk factor framework (Gordy, 2003), the granularity adjustment of Martin & Wilde (2002) and Pykhtin's (2004) multi-factor adjustment. This article aims to complement the existing analytical techniques by considering a fully featured PortfolioManager-type (Kealhofer, 2001) credit portfolio model. Results of numerical tests are presented to demonstrate the performance and prove the validity of the proposed techniques.

First, a short description of the multi-factor Merton-type model is given, followed by a brief review of the progress made so far on the model's analytical tractability. Next, a VAR expansion technique, used as a starting point for the approach presented here, is described. The main results are presented, and a conditional expectation series expansion is derived and applied to systematic risk constituents. Finally, the technique is extended to cover idiosyncratic risk components. Monte Carlo simulations are used to substantiate the validity of the proposed analytical approach.

Structural credit portfolio models

Merton-type credit portfolio models are the most widely accepted ones for credit portfolio risk metrics calculations. In these models, the portfolio consists of risky instruments $\{v_i\}$ with the value v_i of each instrument at the horizon (usually set to one year) being a function of the normally distributed random variable ε_i. Correlations between these variables $\{\varepsilon_i\}$ are modelled through a set of N_f normally distributed independent variables $\{\eta_k\}$ referred to as common factors.[1] Each variable ε_i is split into a sum of an instrument-specific (idiosyncratic) part, which depends on a Gaussian variable ξ_i, and a systematic part as follows:

$$v_i(\varepsilon_i) = v_i\left(\rho_i \sum\nolimits_k (\beta_i)_k \eta_k + \sqrt{1-\rho_i^2}\,\xi_i\right) \tag{1}$$

The independently distributed random variables $\{\{\xi_i\}, \{\eta_k\}\}$ are assumed to have zero mean and unit variance. Instrument-specific constants $|\rho_i| < 1$ and $\{\beta_{ik}\}$ determine the dependency of ε_i on the common factors. The factor loadings $\{\beta_{ik}\}$ are subject to the normalisation condition:

$$\sum\nolimits_k \beta_{ik}^2 = 1 \tag{2}$$

Uncertainty in the value of the portfolio $V = \Sigma_i v_i$ is quantified by means of various risk measures, the most popular of which are VAR, expected shortfall (ES) and standard deviation.

Once the portfolio-level risk measure is known, the question arises of how to allocate this risk consistently among the constituents. The Euler allocation technique (see, for example, Tasche, 2008) is the commonly adopted solution. According to this principle, individual assets v_i of the portfolio are assigned fractions (risk contributions) θ_i of the portfolio-level risk Θ according to:

$$\theta_i = w_i \frac{\partial \Theta}{\partial w_i}, \quad \Theta = \sum_i \theta_i \tag{3}$$

where w_i is a weight of the ith asset in the portfolio. In what follows, to simplify the notation the weights $\{w_i\}$ will not be written explicitly. It will be shown how to generalise Pykhtin's (2004) analytical approach to the credit portfolio risk metrics calculations. Practitioners considering applying Pykhtin's approach to realistic credit portfolio models face two major difficulties. First, Pykhtin's model was formulated for a default-only case and it is not at all obvious how to extend it to a more general and realistic case. Second, calculation of the multi-factor adjustment are of

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[1] *In practice, the common factors correspond to industry and geographic sectors and are not independent. However, their intercorrelation matrix can always be diagonalised and the 'real' common factors can be transformed to independent ones*

quadratic order in portfolio size complexity, making application of the model to large portfolios barely possible. On top of that, no solution to the problem of risk allocation within Pykhtin's model has ever been reported. The approach presented here overcomes the aforementioned difficulties.

VAR and ES adjustments

Building on the work of Gourieroux, Laurent & Scaillet (2000), Martin & Wilde (2002) derived the second-order correction to VAR and used the results in the context of a credit portfolio to calculate an adjustment for undiversified idiosyncratic risk (granularity adjustment). A somewhat simpler derivation is presented here, the outcome of which is a higher-precision correction to VAR and is more suitable for the techniques presented in this article.

Consider a random variable x with a continuous probability density function (PDF) $f(x)$. Let q_α be the α-quantile of this distribution. Consider another random variable δ_x with $g(\delta x|x)$ being its PDF conditional on the value of the first variable x. Let us find the α-quantile q^*_α of the PDF $f^*(x + \delta x)$ of the sum of the above two variables. The f^* can be written as:

$$f^*(x) = \int f(x-\delta x)\, g(\delta x|x-\delta x)\, d(\delta x) \qquad (4)$$

Expanding the right-hand side of this expression in a Taylor series of $(x - \delta x)$ around x, one can obtain:

$$\begin{aligned} f^*(x) &= f(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dx}\left[f(x)\mu_n(x)\right] \\ \mu_n(x) &= \int (\delta x)^n\, g(\delta x|x)\, d(\delta x) \end{aligned} \qquad (5)$$

where $\mu_n(x)$ are the moments of the δx distribution conditional on x.

The relationship between quantiles can be derived by substituting (5) into the following definition of the α-quantile:

$$\int_{-\infty}^{q_\alpha} f(x)\,dx = \alpha = \int_{-\infty}^{q^*_\alpha} f^*(x)\,dx \qquad (6)$$

The result is:

$$\int_{q_\alpha}^{q^*_\alpha} f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{d^{n-1}}{dx^{n-1}}\left[f(x)\mu_n(x)\right]\Big|_{x=q^*_\alpha} \qquad (7)$$

Suppose δx is a small correction to x and assume that $\mu_n \sim \delta^n$, where δ is some small number. One can solve the equation (7) order by order in δ by expanding both sides in powers of $(q^*_\alpha - q_\alpha)$ around q_α.

Without loss of generality, we assume $\mu_1(x) \equiv 0$. In this case, the $\{\mu_n(x)\}$ become conditional central moments and (7) has a particularly simple third-order solution:

$$\begin{aligned} q^*_\alpha - q_\alpha \approx &-\frac{1}{2f(x)} \frac{d}{dx}\left[f(x)\mu_2(x)\right]\Big|_{x=q_\alpha} \\ &+ \frac{1}{6f(x)} \frac{d^2}{dx^2}\left[f(x)\mu_3(x)\right]\Big|_{x=q_\alpha} \end{aligned} \qquad (8)$$

Let us look at the result (8) from a credit portfolio perspective. Let x be a single-factor approximation of the portfolio value, $x = V(\eta)$.[2] Let the factor η be normally distributed with the PDF $n(\eta) = e^{-\eta^2/2}/\sqrt{2\pi}$. The α-quantile q_α is related to the portfolio's VAR and the portfolio's expected value $E(V)$ as[3]:

$$VAR = E(V) - q_\alpha \qquad (9)$$

Using $n'(\eta) = -\eta n(\eta)$, $f(V)dV = n(\eta)d\eta$ and (8), the second- and third-order VAR adjustments can be written as:

$$\begin{aligned} \Delta VAR_2(\alpha) &= \frac{1}{2n(\eta)} \frac{d}{d\eta}\left(\frac{n(\eta)\mu_2(\eta)}{V'(\eta)}\right)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \\ &= \frac{1}{2V'}\left(\mu'_2 - \mu_2\left(\eta + \frac{V''}{V'}\right)\right)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \end{aligned} \qquad (10)$$

$$\begin{aligned} \Delta VAR_3(\alpha) &= \frac{1}{6n(\eta)} \frac{d}{d\eta}\left(\frac{1}{V'(\eta)} \frac{d}{d\eta}\left(\frac{n(\eta)\mu_3(\eta)}{V'(\eta)}\right)\right)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \\ &= \frac{1}{6[V']^2}\Bigg(\mu''_3 - \mu'_3\left(2\eta + 3\frac{V''}{V'}\right) \\ &\quad + \mu_3\left(\left(\eta^2 - 1\right) - 3\eta\frac{V''}{V'} + \frac{3[V'']^2 - V'V'''}{[V']^2}\right)\Bigg)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \end{aligned} \qquad (11)$$

where $\Phi^{-1}(\alpha)$ is the inverse of the normal cumulative PDF.

Using the VAR adjustments (10) and (11), one can easily calculate similar adjustments to ES. Noticing that:

$$ES(\alpha) = \frac{1}{\alpha} \int_{-\infty}^{\eta=\Phi^{-1}(\alpha)} VAR(\eta)\, n(\eta)\, d\eta \qquad (12)$$

the second- and third-order ES contributions can be written as:

$$\Delta ES_2(\alpha) = \frac{1}{2\alpha} \frac{n}{V'} \mu_2\Big|_{\eta=\Phi^{-1}(\alpha)} \qquad (13)$$

$$\begin{aligned} \Delta ES_3(\alpha) &= -\frac{1}{6\alpha} \frac{1}{V'} \frac{d}{d\eta}\left(\frac{n\mu_3}{V'}\right)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \\ &= -\frac{1}{6\alpha} \frac{n}{[V']^2}\left(\mu'_3 - \mu_3\left(\eta + \frac{V''}{V'}\right)\right)\Bigg|_{\eta=\Phi^{-1}(\alpha)} \end{aligned} \qquad (14)$$

Systematic risk

Let us start by assuming that the portfolio dynamics is mainly governed by the systematic risk components, that is, the common factors $\{\eta_k\}$ give the main contribution to the portfolio risk measures, while the idiosyncratic factors $\{\xi_i\}$ give rise to less significant corrections. In this section, it is shown how to isolate the systematic risk by integrating out the idiosyncratic components and how the resulting asymptotic multi-factor framework can be utilised for the risk metrics calculations.

■ **Series expansion for conditional expectation: single factor.** To focus on the systematic part of portfolio dynamics, let us integrate out (average over) the idiosyncratic component ξ_i in (1). Let us assume there is just one common factor and extend the results to a multi-factor case later.

The average value of a facility $\overline{v_i}$ conditional on the systematic factor η is:

$$\overline{v_i}(\eta) = \int v_i\left(\rho_i\eta + \sqrt{1-\rho_i^2}\,\xi\right) \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}\, d\xi \qquad (15)$$

which, after changing the integration variable to:

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[2] *The $V(\eta)$ is assumed to be an invertible function*

[3] *VAR defined this way is simply the economic capital of the portfolio*

$$\varepsilon = \rho_i \eta + \sqrt{1-\rho_i^2}\,\xi$$

becomes:

$$\overline{v_i}(\eta) = \int v(\varepsilon) \frac{1}{\sqrt{1-\rho_i^2}} \exp\left(\frac{2\rho_i \varepsilon_\eta - \rho_i^2 \left(\varepsilon^2 + \eta^2\right)}{2\left(1-\rho_i^2\right)} \right) \frac{e^{-\varepsilon^2/2}}{\sqrt{2\pi}} d\varepsilon \qquad (16)$$

The above expression can be further developed by applying Mehler's formula (for the proof see, for example, Foata, 1978):

$$\sum_{n=0}^{\infty} He_n(\varepsilon) He_n(\eta) \frac{\rho^n}{n!} = \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{2\rho\varepsilon\eta - \rho^2\left(\varepsilon^2+\eta^2\right)}{2\left(1-\rho^2\right)} \right) \qquad (17)$$

where $He_n(\eta) = (-1)^n e^{\eta^2/2} (d/d\eta)^n e^{-\eta^2/2}$ are Hermite polynomials. The result is:

$$\overline{v_i}(\eta) = \sum_n \frac{\rho_i^n}{n!} v_i^{(n)} He_n(\eta), \quad v_i^{(n)} = \int v_i(\varepsilon) He_n(\varepsilon) \frac{e^{-\varepsilon^2/2}}{\sqrt{2\pi}} d\varepsilon \qquad (18)$$

This expansion converges pointwise as long as all the coefficients $v^{(n)}$ are bounded. This is the case, for example, for any piece-wise continuous function $v_i(\varepsilon)$ whose absolute value at infinity does not increase faster than some power of ε. Any reasonable value function of a financial instrument satisfies this constraint.

As a consequence of $|\rho| < 1$ in (18), the conditional expectation series converge significantly better than the Hermite expansion ($\rho = 1$). The conditional expectation function $\overline{v_i}(\eta)$ is not only continuous, but also differentiable an infinite number of times.

The asymptotic single risk factor η value of the portfolio $V_{1f}(\eta) = \Sigma_i \overline{v_i}(\eta)$ can be easily derived from (18) and is:

$$V_{1f}(\eta) = \sum_n V^{(n)} He_n(\eta), \quad V^{(n)} = \sum_i \frac{\rho_i^n}{n!} v_i^{(n)} \qquad (19)$$

Once the coefficients $V^{(n)}$ are calculated, one can immediately write both the VAR and the ES of the portfolio for any confidence level α as:

$$\begin{aligned} VAR(\alpha) &= -\sum_{n>0} V^{(n)} He_n(\eta) \Big|_{\eta=\Phi^{-1}(\alpha)} \\ ES(\alpha) &= \frac{e^{-\eta^2/2}}{\sqrt{2\pi}} \sum_{n>0} V^{(n)} He_{n-1}(\eta) \Big|_{\eta=\Phi^{-1}(\alpha)} \end{aligned} \qquad (20)$$

Using (3) and (19), trivial calculations lead to the following VAR- and ES-based risk contributions:

$$\begin{aligned} VAR_i^c &= -\sum_{n>0} \frac{\rho_i^n}{n!} v^{(n)} He_n(\eta) \Big|_{\eta=\Phi^{-1}(\alpha)} \\ ES_i^c(\alpha) &= \frac{e^{-\eta^2/2}}{\sqrt{2\pi}} \sum_{n>0} \frac{\rho_i^n}{n!} v^{(n)} He_{n-1}(\eta) \Big|_{\eta=\Phi^{-1}(\alpha)} \end{aligned} \qquad (21)$$

■ **Series expansion for conditional expectation: multiple factors.** In a multi-factor case, the conditional expectation (18) can be written as:

$$\overline{v_i}(\eta_k) = \sum_n \frac{\rho_i^n}{n!} v_i^{(n)} He_n\left(\sum\nolimits_k \beta_{ik}\eta_k\right) \qquad (22)$$

This expression, however, does not allow us to write the portfolio value V in a form similar to (19). To accomplish this, let us introduce multivariate Hermite polynomials:

$$\begin{aligned} & He_n^{k_i k_2 \cdots k_n}(\eta_k) \\ & = (-1)^n \exp\left(\frac{1}{2}\sum_m \eta_m^2\right) \frac{\partial}{\partial\eta_{k1}} \frac{\partial}{\partial\eta_{k2}} \cdots \frac{\partial}{\partial\eta_{kn}} \exp\left(-\frac{1}{2}\sum_m \eta_m^2\right) \end{aligned} \qquad (23)$$

The multi-factor expansion then becomes:

$$\overline{v_i}(\eta_k) = \sum_n \frac{\rho_i^n}{n!} v_i^{(n)} \beta_{ik_1}\beta_{ik_2}\cdots\beta_{ik_n} He_n^{k_1 k_2 \cdots k_n}(\eta_k) \qquad (24)$$

and the conditional expectation of the portfolio can be written as:

$$\begin{aligned} V(\eta_k) &= \sum_n \sum_{k_i} V_{k_1 k_2 \cdots k_n}^{(n)} He_n^{k_1 k_2 \cdots k_n}(\eta) \\ V_{k_1 k_2 \cdots k_n}^{(n)} &= \sum_i \frac{\rho_i^n}{n!} v_i^{(n)} \beta_{ik_1}\beta_{ik_2}\cdots\beta_{ik_n} \end{aligned} \qquad (25)$$

The orthogonality property of multivariate Hermite polynomials:

$$\begin{aligned} & \int He_n^{k_1 k_2 \cdots k_n}(\eta_k) He_m^{l_1 l_2 \cdots l_n}(\eta_k) \frac{e^{-\sum_{k=1}^{N_f}\eta_k^2/2}}{(2\pi)^{N_f/2}} d\eta_k \\ & = n!\,\delta_{nm}\delta_{k_1 l_1}\delta_{k_2 l_2}\cdots\delta_{k_n l_n} \end{aligned} \qquad (26)$$

will be used in the following sections.

Using the above, one can calculate the standard deviation of the portfolio, as was recently demonstrated by the author (Voropaev, 2009). From now on, we will focus on the tail risk measures, VAR and ES.

■ **Conditional expectation in the tail.** Let us assume that the portfolio value distribution in the multi-factor case can be approximated by some single-factor value distribution, that is, let us write the value of the portfolio as:

$$V = V_{1f}\left(\vec{Y}\right) + V_{mf}, \quad E\left(V_{mf} \middle| V_{1f}\right) = 0 \qquad (27)$$

where V_{1f} is a single-factor approximation and V_{mf} is a multi-factor correction with zero expectation conditional on V_{1f}. The single systematic risk factor $\vec{Y}$ is a linear combination of the common factors $\{\eta_k\}$. The choice of the principal risk factor $\vec{Y}$ is somewhat arbitrary; however, one would aim to choose $\vec{Y}$ such that V_{1f} is as good an approximation to V as possible and V_{mf} is as small a correction as possible. A solution to this optimisation problem (which needs to be well formulated first) may be a matter of future research. Fortunately, as we will see later, even in the case of a sub-optimal choice of $\vec{Y}$ one can achieve very good numerical results.

The (sub-optimal) choice of $\vec{Y}$ used here is based on the following rationale. Notice that the nth term in the conditional expectation expansion (25) is roughly proportional to ρ_p^n, where ρ_p is some characteristic portfolio correlation. Assuming ρ_p is small, one can conclude that the lower-order terms in (25) give the main contribution to the portfolio dynamics. Assuming further that the $n = 1$ term is the most important one, one would naturally choose $\vec{Y}$ to point in the direction defined by:

$$\vec{V}^{(1)} = \left(V_1^{(1)}, V_2^{(1)}, \cdots, V_{N_f}^{(1)}\right) \qquad (28)$$

This particular choice of $\vec{Y}$ is not only natural and convenient within the proposed framework, but also will be substantiated by

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[4] *Pykhtin (2004) suggests different intuitive choices for $\vec{Y}$. However, they are not theoretically substantiated. In the author's experience, the choice of $\vec{Y}$ presented here leads to better results when applied to realistic portfolios*

numerical tests below.[4] Once the principal risk factor $\vec{Y}$ is known, we can use Gram-Schmidt to ensure it coincides with η_1.

To split the portfolio value (25) according to (27), let us make use of the following identity, which can be derived using the definition of the multivariate Hermite polynomials (23) and the fact that $V^{(n)}_{k_1k_2\cdots k_n}$ are symmetric in $k_1, k_2, \ldots, k_n$:

$$V^{(n)}_{k_1\cdots k_n} He_n^{k_1\cdots k_n}(\eta_k) = \sum_{l=0}^{n} V^{(n)}_{\underbrace{11\cdots1}_{n-l}k_1k_2\cdots k_n} \binom{n}{l} He_{n-l}(\eta_1) He_l(\eta^*_k) \quad (29)$$

where $\binom{n}{l}$ are binomial coefficients and η^*_k is a set of all common factors but η_1. Using the above expression, the portfolio value (25) can be written as:

$$V(\eta_k) = \sum_n \sum_{k_i} \sum_{m\geq n} \binom{m}{n} He_{m-n}(\eta_1) V^{(m)}_{\underbrace{11\cdots1}_{m-n}k_1k_2\cdots k_n} He_n^{k_1k_2\cdots k_n}(\eta^*_k) \quad (30)$$

Finally, separating the $n = 0$ term and introducing conditional coefficients $V^{(n)}_{mfk_1k_2\ldots k_n}(\eta_1)$, the portfolio value can be put into the form:

$$V(\eta_k) = V_{1f}(\eta_1) + V_{mf}(\eta^*_k|\eta_1) \quad (31)$$

$$\begin{aligned} V_{1f}(\eta_1) &= \sum_n V^{(n)}_{1f} He_n(\eta_1) \\ V_{mf}(\eta^*_k|\eta_1) &= \sum_{n>0}\sum_{k_i} V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1) He_n^{k_1k_2\cdots k_n}(\eta^*_k) \end{aligned} \quad (32)$$

$$\begin{aligned} V^{(n)}_{1f} &= V^{(n)}_{\underbrace{11\cdots1}_{n}} \\ V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1) &= \sum_{m\geq n} \binom{m}{n} He_{m-n}(\eta_1) V^{(m)}_{\underbrace{11\cdots1}_{m-n}k_1k_2\cdots k_n} \end{aligned} \quad (33)$$

The multi-factor correction V_{mf} in the above has zero expectation conditional on η_1 due to the orthogonality properties (26). For a given confidence level α, the above expressions represent series expansion of the conditional (on $\eta_1 = \Phi^{-1}(\alpha)$) tail expectation.

■ **Systematic tail risk and its allocation.** The series expansion of the conditional tail expectation (31), (32) and (33) together with the single-factor case results (20) allow us to apply the results of the previous section to VAR and ES calculations.

Since the single-factor VAR and ES have been calculated before, that is, (20), let us start with the second-order contributions (10) and (13). Using the notations introduced in the previous section, the second-order VAR and ES adjustments are:

$$\begin{aligned} &\Delta VAR_2(\alpha) \\ &= \frac{1}{2V'_{1f}(\eta_1)}\left(\mu'_2(\eta_1) - \mu_2(\eta_1)\left(\eta_1 + \frac{V''_{1f}(\eta_1)}{V'_{1f}(\eta_1)}\right)\right)\Bigg|_{\eta_1=\Phi^{-1}(\alpha)} \\ &\Delta ES_2(\alpha) = \frac{1}{2\alpha}\frac{n(\eta_1)}{V'_{1f}(\eta_1)}\mu_2(\eta_1)\Bigg|_{\eta_1=\Phi^{-1}(\alpha)} \end{aligned} \quad (34)$$

The V_{1f} derivatives can be calculated using (32) and are:

$$\begin{aligned} V'_{1f}(\eta_1) &= \sum_{n>0} V^{(n)}_{1f} n He_{n-1}(\eta_1) \\ V''_{1f}(\eta_1) &= \sum_{n>1} V^{(n)}_{1f} n(n-1) He_{n-2}(\eta_1) \end{aligned} \quad (35)$$

The conditional second central moment (variance) $\mu_2(\eta_1)$ is:

$$\mu_2(\eta_1) = \sum_{n>0} n! \sum_{k_i} \left[V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1)\right]^2 \quad (36)$$

and its derivative $\mu'_2(\eta_1)$ can be calculated as:

$$\mu'_2(\eta_1) = 2\sum_{n>0} n! \sum_{k_i} V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1)\left[V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1)\right]' \quad (37)$$

where:

$$\left[V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1)\right]' = \sum_{m>n}\binom{m}{n}(m-n) He_{m-n-1}(\eta_1) V^{(m)}_{11\cdots1k_1k_2\cdots k_n} \quad (38)$$

The corresponding risk contributions can be calculated by applying (3) to (34). The following identities facilitate the exercise:

$$\begin{aligned} w_i\frac{\partial}{\partial w_i}V'_{1f}(\eta_1) &= \sum_{n>0}\frac{\rho_i^n}{n!}v_i^{(n)}(\beta_i)_1^n n He_{n-1}(\eta_1) \\ w_i\frac{\partial}{\partial w_i}\mu_2(\eta_1) &= 2\sum_{n>0}\rho_i^n\sum_{k_i}V^{(n)}_{mf_{k_1k_2\cdots k_n}}(\eta_1) \\ &\quad + \sum_{m\geq n}\binom{m}{n} He_{m-n}(\eta_1) v_i^{(m)} \beta_{i\,1}^{\,m-n}\beta_{ik_1}\beta_{1k_2}\cdots\beta_{ik_{m-n}} \end{aligned} \quad (39)$$

Calculations of the third-order VAR and ES adjustments, (11) and (14), and corresponding risk contributions can be done in the same fashion. The difficulty one will face in this case is calculating $\mu_3(\eta_1)$. To calculate the third central moment, the following integral has to be evaluated:

$$\mu_3(\eta_1) = \int\left[V_{mf}(\eta^*_k|\eta_1)\right]^3 \frac{e^{-\sum_{k=2}^{N_f}\eta_k^2/2}}{(2\pi)^{(N_f-1)/2}}d\eta^*_k \quad (40)$$

Unlike the case of $\mu_2(\eta_1)$, orthogonality conditions (26) alone are not sufficient to calculate the integral. One faces the problem of calculating the exponentially weighted average of three Hermite polynomials. To solve this problem, let us start with the following identity (which follows from a more general result of Drake, 2009):

$$He_n(x)He_m(x) = \sum_k \binom{n}{k}\binom{m}{k} k! He_{n+m-2k}(x) \quad (41)$$

The integral then can be solved as follows:

$$\int dx He_n(x)He_m(x)He_k(x)\frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{n!m!k!}{\left(\frac{m+k-n}{2}\right)!\left(\frac{k+n-m}{2}\right)!\left(\frac{n+m-k}{2}\right)!} \quad (42)$$

provided $m + n + k$ is even and each of m, n, k does not exceed the sum and is not less than the absolute value of the other two. Otherwise, the integral is zero.

It is not clear how to write a multivariate version of the above identities. However, using the above identities together with the definition of multivariate Hermite polynomials (23), one can solve for any given set of n, m, k. For example:

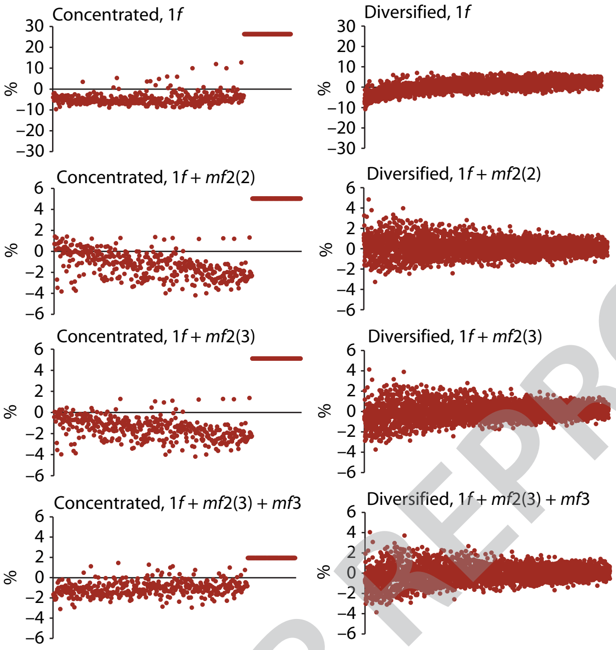
$$\begin{aligned} &\int He_1^{k_1}(\eta^*_k) He_2^{k_2k_3} He_3^{k_4k_5k_6}(\eta^*_k)\frac{e^{-\sum_{K=2}^{N_f}\eta_k^2/2}}{(2\pi)^{(N_f-1)/2}}d\eta^*_k \\ &= 6\delta_{k_1k_4}\delta_{k_2k_5}\delta_{k_3k_6} \end{aligned} \quad (43)$$

The first few terms of the third central moment $\mu_3(\eta_1)$ are:

A. Relative differences between analytical approximation and Monte Carlo simulation on portfolio level

	1*f*	1*f* + *mf*2(2)	1*f* + *mf*2(3)	1*f* + *mf*2(3) + *mf*3
Concentrated	–5.2%	–0.9%	–0.8%	–0.1%
Diversified	–1.5%	–0.1%	–0.1%	0.0%

1 Relative differences between Monte Carlo and analytical estimates of the systematic VAR-based risk contributions



$$\mu_3(\eta_1) = 2\sum_{k_1,k_2} V^{(1)}_{mf_{k_1}}(\eta_1)V^{(1)}_{mf_{k_2}}(\eta_1)V^{(2)}_{mf_{k_1k_2}}(\eta_1) + 6\sum_{k_1,k_2,k_3} V^{(1)}_{mf_{k_1}}(\eta_1)V^{(2)}_{mf_{k_2k_3}}(\eta_1)V^{(3)}_{mf_{k_1k_2k_3}}(\eta_1) + 8\sum_{k_1,k_2,k_3} V^{(2)}_{mf_{k_1k_2}}(\eta_1)V^{(2)}_{mf_{k_2k_3}}(\eta_1)V^{(2)}_{mf_{k_3k_1}}(\eta_1) + \cdots \quad (44)$$

The results presented in this section allow us to calculate portfolio-level and facility-level systematic components of VAR and ES. It is easy to see that the necessary amount of calculations is linear in a number of facilities of the portfolio. Moreover, the calculations can easily be parallelised on multi-processor machines.

■ **Numerical results.** To prove the validity and demonstrate the accuracy of the proposed analytical framework, let us compare results of the analytical approximation with those of unbiased Monte Carlo simulation. The focus here will be on VAR and VAR-based risk contributions. Let us first limit the analysis to the systematic risk components, which is achieved as follows. For each scenario, a set of systematic factors is generated. Instead of generating borrower-specific factors, however, expected (given systematic factors) values are assigned per facility.

The particular set of common factors used in the tests is similar to the one described in Kealhofer (2001). The total of $N_f = 120$ factors cover 61 industry and 45 regional sectors. Two portfolios were constructed, diversified and concentrated. Both portfolios contain identical loans maturing at the horizon. Each loan's correlation with the systematic factors ρ_i is 0.6 and probability of default equals 1%. The corresponding value function $v_i(\varepsilon)$ is:

$$v_i(\varepsilon) = \begin{cases} 1 & if \quad \varepsilon > \Phi^{-1}(0.01) \\ 0 & if \quad \varepsilon \le \Phi^{-1}(0.01) \end{cases} \quad (45)$$

The diversified portfolio contains 45 × 61 = 2,745 loans, each loan representing a different region/industry. The concentrated portfolio contains 400 loans randomly assigned to a different region/industry and 100 loans representing a single region/industry pair. These 100 loans create region/industry concentration in the portfolio.

Monte Carlo estimates of portfolio VAR and VAR contributions per facility were based on 1 billion scenarios. The confidence interval was set to 99.9%. Estimates of VAR contributions were calculated based on 500,000 scenarios around the 99.9% point (that is, average VAR contributions for the 99.875–99.925% interval were calculated). Plain vanilla Monte Carlo simulations were used to exclude any bias and limit possibilities of implementation errors.

Several analytical estimates were calculated. First, a single-factor approximation (1*f*) was calculated based on (31), (32) and (33), and (20) and (21). Next, second-order (multi-factor) VAR adjustment (10) was added. The second central moment μ_2 used for calculations was computed using the first two (1*f* + *mf*2(2)) and three (1*f* + *mf*2(3)) terms in its series expansion (36). Finally, analytical estimates were completed by the third-order (1*f* + *mf*2(3) + *mf*3) VAR adjustment (11). The estimation of the third central moment μ_3 was based on the first three terms of its series expansion listed in (44).

Comparison of the portfolio level results is presented in table A, while VAR-based risk contributions at facility level are compared in figure 1. The results presented in figure 1 are sorted increasing from left to right order using a scalar product of the principal vector $\vec{Y}$ and factor loadings vectors $\vec{\beta}$ as a parameter.

The following conclusions can be drawn based on the results of the numerical tests. Overall, the analytical approximation produces excellent results. On a portfolio level, a slight underestimation of VAR (economic capital) by a single-factor approximation is observed for concentrated portfolios. The situation is improved by higher-order corrections, whose contributions lead to very precise results. In the case of VAR contributions, the higher-order corrections to the single-factor approximation (second- and third-order VAR adjustments) are necessary to achieve high accuracy. The resulting analytical estimates of the VAR contributions are just 1–2% different from the Monte Carlo-based estimates.

Idiosyncratic risk

The asymptotic multi-factor framework described in the previous sections was built on the conditional expectation series expansion. As a result, the idiosyncratic risk of the portfolio has been wiped out (averaged over) and portfolio risk measures were expressed in terms of the systematic components. In this section, it is shown how the framework described so far can be extended to cover the idiosyncratic risk components.

■ **Idiosyncratic contributions.** Using the notations of the previous section and introducing the idiosyncratic value, or granularity adjustment (ga), component V_{ga}, the full portfolio value V can be written as:

$$V\left(\eta_1,\eta^*,\xi\right) = V_{1f}\left(\eta_1\right) + V_{mf}\left(\eta^*\middle|\eta_1\right) + V_{ga}\left(\xi\middle|\eta_1,\eta^*\right) \quad (46)$$

where:

$$V_{1f}\left(\eta_1\right)=\left\langle V\left(\eta_1,\eta^*,\xi\right)\right\rangle_{\eta^*,\xi} \tag{47}$$

$$V_{mf}\left(\eta^*\middle|\eta_1\right)=\left\langle V\left(\eta_1,\eta^*,\xi\right)\right\rangle_{\xi}-\left\langle V\left(\eta_1,\eta^*,\xi\right)\right\rangle_{\eta^*,\xi} \tag{48}$$

$$V_{ga}\left(\xi\middle|\eta_1,\eta^*\right)=V\left(\eta_1,\eta^*,\xi\right)-\left\langle V\left(\eta_1,\eta^*,\xi\right)\right\rangle_{\xi} \tag{49}$$

The $\langle\ldots\rangle$ in the above stands for average. Also, by construction:

$$\left\langle V_{ga}\right\rangle_{\xi}=0,\quad \left\langle V_{mf}\right\rangle_{\eta^*}=0 \tag{50}$$

Following Pykhtin's (2004) approach, one can treat the sum $V_{mf}+V_{ga}$ as a small correction to V_{1f} and apply the results of the adjustments section above ('VAR and ES adjustments'). The VAR and ES contributions are then expressed in terms of V_{1f} and central moments of $V_{mf}+V_{ga}$. For the second and third central moments considered here, one can write the following:

$$\mu_2\left[V_{mf}+V_{ga}\right]=\mu_2\left[V_{mf}\right]+\left\langle\mu_2\left[V_{ga}\left(\xi\right)\right]\right\rangle_{\eta^*} \tag{51}$$

$$\begin{aligned}&\mu_3\left[V_{mf}+V_{ga}\right]\\&=\mu_3\left[V_{mf}\right]+3\left\langle V_{mf}\cdot\mu_2\left[V_{ga}\left(\xi\right)\right]\right\rangle_{\eta^*}+\left\langle\mu_3\left[V_{ga}\left(\xi\right)\right]\right\rangle_{\eta^*}\end{aligned} \tag{52}$$

The first of the above is a well-known law of total variance, while the second follows from a more general law of total cumulance.

Thus, to calculate the idiosyncratic contribution to the second and the third VAR and ES adjustments, one needs to calculate the second and the third conditional central moments of the idiosyncratic component V_{ga}. Only these moments, not detailed information about idiosyncratic components, are needed to complete the task.

Conditional idiosyncratic moments. Taking advantage of the conditional (on systematic factors) independence of the idiosyncratic components, the second and the third idiosyncratic central moments can be calculated as a simple sum of individual (from each loan in the portfolio) contributions. The conditional expectation series expansion (24) can be applied not just to the value, but to its powers:

$$\overline{v_i}\left(\eta\right)=\sum_n\frac{\rho_i^n}{n!}v_i^{(n)}\beta_{ik_1}\beta_{ik_2}\cdots\beta_{ik_n}He_n^{k_1k_2\cdots k_n}\left(\eta_k\right) \tag{53}$$

$$\overline{v_i^2}\left(\eta\right)=\sum_n\frac{\rho_i^n}{n!}w_i^{(n)}\beta_{ik_1}\beta_{ik_2}\cdots\beta_{ik_n}He_n^{k_1k_2\cdots k_n}\left(\eta_k\right) \tag{54}$$

$$\overline{v_i^3}\left(\eta\right)=\sum_n\frac{\rho_i^n}{n!}u_i^{(n)}\beta_{ik_1}\beta_{ik_2}\cdots\beta_{ik_n}He_n^{k_1k_2\cdots k_n}\left(\eta_k\right) \tag{55}$$

where:

$$\begin{aligned}v_i^{(n)}&=\int v_i\left(\varepsilon\right)He_n\left(\varepsilon\right)\frac{e^{-e^2/2}}{\sqrt{2\pi}}d\varepsilon\\w_i^{(n)}&=\int v_i^2\left(\varepsilon\right)He_n\left(\varepsilon\right)\frac{e^{-e^2/2}}{\sqrt{2\pi}}d\varepsilon\\u_i^{(n)}&=\int v_i^3\left(\varepsilon\right)He_n\left(\varepsilon\right)\frac{e^{-e^2/2}}{\sqrt{2\pi}}d\varepsilon\end{aligned} \tag{56}$$

The above expansions can be used to calculate the conditional central moments of the ith loan in the portfolio:

$$\left\langle\left(\mu_2\right)_i\right\rangle_{\eta^*}=\left\langle\overline{v_i^2}\right\rangle_{\eta^*}-\left\langle\overline{v_i}^2\right\rangle_{\eta^*} \tag{57}$$

$$\left\langle\left(\mu_3\right)_i\right\rangle_{\eta^*}=\left\langle\overline{v_i^3}\right\rangle_{\eta^*}-3\left\langle\overline{v_i}\cdot\overline{v_i^2}\right\rangle_{\eta^*}+2\left\langle\overline{v_i}^3\right\rangle_{\eta^*} \tag{58}$$

and the mixed term in (52):

$$3\left\langle V_{mf}\cdot\left(\mu_2\right)_i\right\rangle_{\eta^*}=3\left\langle V_{mf}\cdot\overline{v_i^2}\right\rangle_{\eta^*}-3\left\langle V_{mf}\cdot\overline{v_i}^2\right\rangle_{\eta^*} \tag{59}$$

The averages $\langle\ldots\rangle_{\eta^*}$ can be calculated using the techniques developed in the previous section. For example:

$$\left\langle\overline{v_i^3}\right\rangle_{\eta^*}=\sum_n\frac{\beta_1^n}{n!}u^{(n)}He_n\left(\eta_1\right) \tag{60}$$

$$\begin{aligned}\left\langle\overline{v_i}\cdot\overline{v_i^2}\right\rangle_{\eta^*}=\sum_{k=0}^{\infty}k!&\left(\sum_{n=k}^{\infty}\binom{n}{k}v^{(n)}He_{n-k}\left(\eta_1\right)\left(\beta_1\right)^{n-k}\left(\sqrt{1-\beta_1^2}\right)^k\right)\\&\times\left(\sum_{n=k}^{\infty}\binom{n}{k}w^{(n)}He_{n-k}\left(\eta_1\right)\left(\beta_1\right)^{n-k}\left(\sqrt{1-\beta_1^2}\right)^k\right)\end{aligned} \tag{61}$$

$$\left\langle\overline{v_i}^3\right\rangle_{\eta^*}=\sum_{k,l,m}\frac{k!l!m!}{\left(\frac{l+m-k}{2}\right)!\left(\frac{m+k-l}{2}\right)!\left(\frac{k+l-m}{2}\right)!}h_kh_lh_m \tag{62}$$

where:

$$h_k=\sum_{n=k}^{\infty}\binom{n}{k}v^{(n)}He_{n-k}\left(\eta_1\right)\left(\beta_1\right)^{n-k}\left(\sqrt{1-\beta_1^2}\right)^k \tag{63}$$

The central moments μ_2 and μ_3 of the portfolio are calculated by summing over the $\langle(\mu_2)_i\rangle_{\eta^*}$ and $\langle(\mu_3)_i\rangle_{\eta^*}$ in (57) and (58). The derivatives of these central moments are trivially calculated in a similar fashion using the property $He_n'(\eta_1)=n\cdot He_{n-1}(\eta_1)$. Once the central moments and their derivatives have been calculated, the VAR and ES adjustments are obtained using (10) and (11), and (13) and (14).

Allocation of idiosyncratic risk is a straightforward (although somewhat laborious) task. The Euler principle (3) can be applied to the idiosyncratic risk component in a way similar to the one described above ('Systematic tail risk and its allocation'). Due to the conditional independence, the idiosyncratic contribution is a sum of contributions from individual loans. Hence, the amount of calculations needed is linear in a number of loans in the portfolio.

However, these individual contributions are not equal to the risk contributions calculated using the Euler allocation principle. This is because the partial derivative in $w_i(\partial/\partial w_i)$ is applied not only to the central moments (and their derivatives), but also to the V', V'', ... terms in (10) and (11), and (13) and (14). For example, a loan with $\rho_i = 1$ in (1), that is, with the value depending on systematic risk factors only, gives a zero contribution to the portfolio idiosyncratic moments μ_2 and μ_3, but its contribution may be significant.

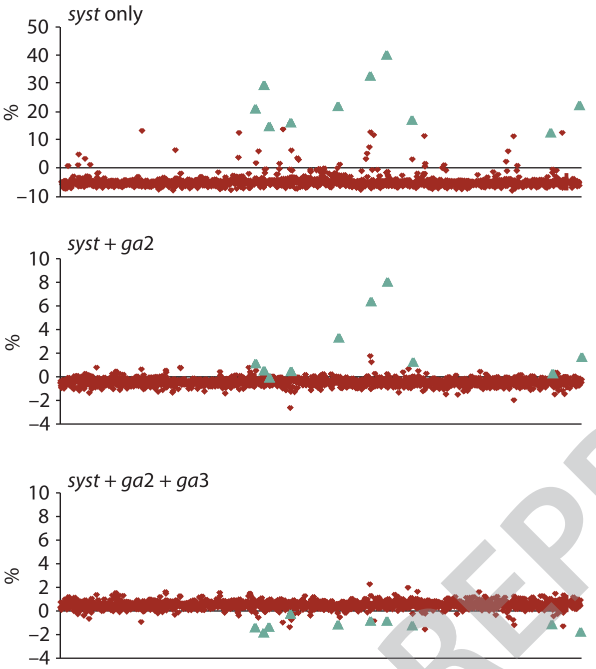
Numerical results. The analysis presented here mainly focuses on the effects of the idiosyncratic risk and its concentrations. In contrast with the numerical results above, a realistic portfolio was used to benchmark the proposed analytical techniques against Monte Carlo simulations. As before, the portfolio VAR and the marginal VAR contributions were the risk measures of interest.

The portfolio consisted of 2,000 loans to distinct customers randomly selected from a loan portfolio of a large European bank. The set of common systematic factors covering 45 geographic regions and 61 regions, as well as the valuation function at the horizon $v_i(\varepsilon)$ used in the experiment were similar to those

B. Relative differences between analytical and simulation-based estimates of portfolio VAR

syst	*syst + ga2*	*syst + ga2 + ga3*
–4.1%	–0.3%	–0.1%

2 Relative differences between Monte Carlo and analytical estimates of the VAR-based risk contributions



of the PortfolioManager (Kealhofer, 2001) model.

Both the portfolio VAR and VAR contributions were estimated using unbiased Monte Carlo simulations. The confidence level was set at 99.9%. To reduce simulation noise, the VAR contributions were estimated as value expectations in the interval 99.85–99.95%. The simulations consisted of 10^{10} (10 billion) scenarios.

The analytical estimates used for comparison with the simulation-based ones were constructed as follows. The systematic part of VAR (*syst*) was used as a starting point. These estimates covered up to the third-order VAR adjustment as described in the earlier numerical results. The second and third estimates were calculated by consequently adding the second (*ga*2) and the third (*ga*3) order[5] idiosyncratic VAR adjustments to the systematic part.

The relative differences in the portfolio VARs are presented in table B. The initial underestimation of VAR by the analytical approximation of systematic risk only is improved significantly by taking into account the second- and third-order idiosyncratic contributions. Even without the third-order corrections, the portfolio-level results are accurate enough for any practical purposes.

The situation is different if we compare the marginal VAR contributions. Figure 2 shows the relative differences in the VAR contributions. The triangles correspond to the 10 loans with the highest VAR contributions. These are the loans with the highest single-name risk concentration in the portfolio. The top scatter plot in figure 2 shows that significant differences between the systematic (analytical) and the full (simulation-based) VAR contributions exist for the concentrated exposures. Both the second and the third idiosyncratic corrections need to be taken into account to obtain precise estimates.

While a few of the biggest loans (in terms of VAR contributions) receive significant positive contributions to their VARs due to idiosyncratic risk, most loans receive small negative contributions.

Applying the second and the third VAR adjustments to both systematic and idiosyncratic risk components leads to excellent results. At the portfolio level, the results may be considered as exact. Calculation of the marginal VAR contributions is a more challenging task. However, very high accuracy is achieved by taking into account higher-order (the third order in this case) VAR adjustments.

Conclusion

The analytical framework for structured credit portfolio models presented here is an extension and improvement of the one developed by Pykhtin (2004). Second- and third-order VAR and ES adjustments were considered. The default-only case considered by Pykhtin was extended to the case of arbitrary valuation function at the horizon. The problem of quadratic (in portfolio size) complexity of Pykhtin's multi-factor adjustment has been solved. High accuracy of the proposed technique was demonstrated by benchmarking with Monte Carlo simulations. The realised performance of the analytical approximation allows it to be considered as not just a supplement, but a substitute to the conventional simulation-based calculations. ■

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[5] *The mixed term in (52) was included in the third-order idiosyncratic contribution. This term gave a negligible contribution in the test presented. However, it is, not clear if the term can be neglected in a general case*