Calibration of CDO tranches with the dynamical GPL model

Consistent calibration of a credit index and its tranches across maturities with a single arbitrage-free model is a difficult problem. Here, ***Damiano Brigo****,* ***Andrea Pallavicini*** *and* ***Roberto Torresetti*** *show that a simple loss dynamics based on the generalised Poisson process achieves good results. This model allows for generalisations, achieving stochastic spread dynamics and consistency with single names*

We consider a dynamical model for the loss distribution of a pool of names. Our model focuses on three points: tractability – the loss distribution should be known analytically; the calibration of market information, currently quoted index collateralised debt obligation (CDO) tranches and tranchelets for several maturities should be possible, and realistic numerical examples should be given; and the pricing of correlation products depending on the loss distribution dynamics should be feasible in a reasonable amount of time, possibly by simulation.

The basic idea of our approach consists of modelling the process driving the pool loss as a linear combination of independent Poisson processes with different intensities or, in other words, as a generalised Poisson process. The jumps of different processes are multiplied by different positive amplitudes representing losses of different sizes. Each loss size is ideally associated with the instantaneous default of a specific number of names in the pool. Default self-exciting features of the model are embedded by allowing for more than one default in short time intervals. We may calibrate to market data both the loss jump sizes and the intensities in the single Poisson processes driving the jumps of each possible loss size. The resulting model is called the generalised Poisson loss (GPL) dynamical model.

Different frameworks with loss dynamics have been proposed recently, and we provide references and comments on the relationships with our approach in Brigo, Pallavicini & Torresetti (2006). Here, we aim at a completely specified and manageable model, rather than at an abstract framework, and we present detailed calibration results, with numerical outputs for the DJ iTraxx index pool. Results show the calibration to be good and the model parameters to be relatively stable over time.

Modelling the aggregate loss directly rather than obtaining it from single default models with a dependence structure constitutes the 'top-down' approach. Often this approach leads to difficulties in achieving full consistency with single names. Instead, our approach can be generalised into a default-cluster-based model, called the generalised Poisson cluster loss (GPCL) model. This generalises the earlier results of Lindskog & McNeil (2003) and Elouerkhaoui (2006) on common Poisson shock models. This leads to rigorous and clear consistency with single names, as we point out in Brigo, Pallavicini & Torresetti (2007), where we also show calibration results for the CDX index pool. Finally, an extended discussion of the model presented here, with further calibration examples involving tranchelets, and with enriched and tractable stochastic spread and recovery dynamics, can be found in Brigo, Pallavicini & Torresetti (2006). These extensions will prove their possible use and be testable when liquid quotes for tranche options and forward-starting tranches are available.

Market quotes

The most liquid multi-name credit instruments available in the market are credit indexes and CDO tranches (for example, DJ iTraxx and CDX). We discuss them in the following.

■ **Credit indexes.** The index is given by a pool of names 1, 2, ... , M, where typically $M = 125$, each with notional $1/M$ so that the total pool has unitary notional. The index default leg consists of protection payments corresponding to the defaulted names of the pool. Each time one or more names default, the corresponding loss increment is paid to the protection buyer, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

In exchange for loss increase payments, a periodic premium with rate S is paid from the protection buyer to the protection seller, until final maturity T_b. This premium is calculated on a notional that decreases each time a name in the pool defaults, and decreases by an amount corresponding to the notional of that name (without taking out the recovery).

We denote by $\bar{L}_t$ the portfolio cumulated loss and by $\bar{C}_t$ the number of defaulted names up to time t divided by M. Since at each default part of the defaulted notional is recovered, we have $0 \le \bar{L}_t \le \bar{C}_t \le 1$. The discounted payout of the two legs of the index is given as follows:

$$DefLeg(0) := \int_0^T D(0,t)\,d\bar{L}_t$$

$$PremiumLeg(0) = S_0 \sum_{i=1}^{b} \delta_i D(0,T_i)\left(1-\bar{C}_{T_i}\right)$$

where $D(s, t)$ is the discount factor (often assumed to be deterministic) between times s and t, and $\delta_i = T_i - T_{i-1}$ is the year fraction. In the second equation, the actual outstanding notional in each period would be an average over $[T_{i-1}, T_i]$, but we replaced it with the value of the outstanding notional at T_i for simplicity, as is commonly done.

Notice that, in contrast to what will happen with the tranches (see the following section), here the recovery is not considered when calculating the outstanding notional, in that only the number of defaults matters.

The market quotes the values of S_0 that, for different maturities, balance the two legs. If one has a model for the loss and the number of defaults, one may make the loss and number of defaults in the model, when plugged inside the two legs, lead to the same risk-neutral expectation (and thus price) when the quoted S_0 is inside the premium leg, leading to:

$$S_0 = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)\, d\bar{L}_t\right]}{\mathbb{E}_0\left[\sum_{i=1}^{b} \delta_i D(0,T_i)\left(1-\bar{C}_{T_i}\right)\right]} \tag{1}$$

CDO tranches. Synthetic CDOs with maturity T are contracts involving a protection buyer, a protection seller and an underlying pool of names. They are obtained by putting together a collection of credit default swaps (CDSs) with the same maturity on different names, 1, 2, ... , M, where typically $M = 125$, each with notional $1/M$, and then 'tranching' the loss of the resulting pool between the points A and B, with $0 \le A < B \le 1$:

$$\bar{L}_t^{A,B} := \frac{1}{B-A}\left[\left(\bar{L}_t - A\right)1_{\{A<\bar{L}_t\le B\}} + \left(B-A\right)1_{\{\bar{L}_t > B\}}\right]$$

Once enough names have defaulted and the loss has reached A, the count starts. Each time the loss increases, the corresponding loss change rescaled by the tranche thickness $B - A$ is paid to the protection buyer, until maturity arrives or until the total pool loss exceeds B, in which case the payments stop.

The discounted default leg payout can then be written as:

$$DefLeg(0;A,B) := \int_0^T D(0,t)\, d\bar{L}_t^{A,B}$$

One should not be confused by the integral, because the loss $\bar{L}_t^{A,B}$ changes with discrete jumps. Analogously, the total loss $\bar{L}_t$ and the tranche's outstanding notional change with discrete jumps.

As usual, in exchange for the protection payments, a premium rate $S_0^{A,B}$, fixed at time $T_0 = 0$, is paid periodically, say at times T_1, T_2, ... , $T_b = T$. Part of the premium can be paid at time $T_0 = 0$ as an upfront $U_0^{A,B}$. The rate is paid on the 'survived' average tranche notional. If we further assume payments are made on the notional remaining at each payment date T_i, rather than on the average in $[T_{i-1}, T_i]$, the premium leg payout can be written as:

$$PremiumLeg(0;A,B) := U_0^{A,B} + S_0^{A,B}\sum_{i=1}^{b}\delta_i D(0,T_i)\left(1-\bar{L}_{T_i}^{A,B}\right)$$

When pricing CDO tranches, one is interested in the premium rate $S_0^{A,B}$ that sets the risk-neutral price of the tranche to zero. The tranche value is calculated taking the (risk-neutral) expectation (in $t = 0$) of the discounted payout consisting of the difference between the default and premium legs above. We obtain:

$$S_0^{A,B} = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)\, d\bar{L}_t^{A,B}\right] - U_0^{A,B}}{\mathbb{E}_0\left[\sum_{i=1}^{b}\delta_i D(0,T_i)\left(1-\bar{L}_{T_i}^{A,B}\right)\right]} \tag{2}$$

The above expression can be easily recast in terms of the upfront premium $U_0^{A,B}$ for tranches that are quoted in terms of upfront fees.

The tranches that are quoted on the market refer to standardised pools, standardised attachment-detachment points $A - B$ and standardised maturities T.

Actually, for the DJ iTraxx and CDX pools, the equity tranche ($A = 0$, $B = 3\%$) is quoted by means of the fair $U_0^{A,B}$, while assuming $S_0^{A,B} = 500$ basis points. All other tranches are quoted by means of the fair $S_0^{A,B}$, assuming no upfront fee ($U_0^{A,B} = 0$).

Model definition

The no-arbitrage expressions for the quoted spread of credit indexes, given by equation (1), and of CDO tranches, given by equation (2), show that the only information we can infer from market quotes are expected quantities, while we lack direct information about dependencies across single names. In particular, credit indexes depend both on expected portfolio cumulated loss and on expected number of defaults, while CDO tranches depend only on expected tranched portfolio cumulated loss.

This market data suggests modelling loss-related quantities, that is, portfolio cumulated loss and number of defaults, directly as fundamental objects, rather than patching single default models through a (static) copula.

The underlying GPL dynamics. The GPL model can be formulated as follows. Consider a probability space supporting a number n of independent Poisson processes N_1, ... , N_n with time-varying, and possibly stochastic, intensities λ_1, ... , λ_n under the risk-neutral measure $\mathbb{Q}$. The risk-neutral expectation conditional on the market information up to time t, including the pool loss evolution up to t, is denoted by $\mathbb{E}_t$. Intensities, if stochastic, are assumed to be adapted to such information.

Define the stochastic process:

$$Z_t := \sum_{j=1}^{n} \alpha_j N_j(t) \tag{3}$$

for positive integers $\alpha_1, \dots, \alpha_n$. In the following, we refer to the Z_t process simply as the GPL process. We will use this process as a driving process for the cumulated portfolio loss ($\bar{L}_t$), the relevant quantity for our payouts. In Brigo, Pallavicini & Torresetti (2006), the possible use of the GPL process as a driving tool for the default counting process $\bar{C}_t$ is illustrated instead.

The characteristic function of the Z_t process is:

$$\varphi_{Z_t}(u) = \mathbb{E}_0\left[e^{iuZ_t}\right] = \mathbb{E}_0\left[\mathbb{E}_0\left[e^{iuZ_t} \middle| \Lambda_1(t)\dots\Lambda_n(t)\right]\right]$$

where $\Lambda_i(t) := \int_0^t \lambda_j(s)ds$, with $i = 1 \dots n$, are the cumulated intensities of each Poisson process. Now, we substitute Z_t obtaining:

$$\begin{aligned}\varphi_{Z_t}(u) &= \mathbb{E}_0\left[\prod_{j=1}^{n} \mathbb{E}_0\left[e^{iu\alpha_j N_j(t)} \middle| \Lambda_1(t)\dots\Lambda_n(t)\right]\right] \\ &= \mathbb{E}_0\left[\prod_{j=1}^{n} \varphi_{N_j(t)|\Lambda_j(t)}(u\alpha_j)\right]\end{aligned}$$

which can be directly calculated since the characteristic function $\varphi_{N_j(t)|\Lambda_j(t)}$ of each Poisson process, given its intensity, is known in closed form, leading to:

$$\varphi_{Z_t}(u) = \mathbb{E}_0\left[\exp\left(\sum_{j=1}^{n} \Lambda_j(t)\left(1 - e^{iu\alpha_j}\right)\right)\right] \tag{4}$$

The marginal distribution p_{Z_t} of the process Z_t can be directly calculated at any time via inverse Fourier transformation of the characteristic function of the process. The characteristic function $\varphi_{Z_t}(u)$ can be explicitly calculated for some relevant choices of Poisson cumulated intensities distributions (see, for example, Brigo, Pallavicini & Torresetti, 2006).

Equivalent formulation as compound Poisson process. One more way of looking at our process is as a compound Poisson process. Indeed, at any time t our process Z_t has the same characteristic function as a particular compound Poisson process. Consider the following compound Poisson process:

$$X_t = \sum_{j=1}^{N_t} Y_j$$

where N is a standard Poisson process with intensity λ and the Y_js are independent and identically distributed random variables, independent of N, and with distribution given by:

$$Y_j \sim \begin{cases} \alpha_1 & \lambda_1 / \left(\sum_{k=1}^{n} \lambda_k\right) \\ \alpha_2 & \lambda_2 / \left(\sum_{k=1}^{n} \lambda_k\right) \\ \vdots & \\ \alpha_n & \lambda_n / \left(\sum_{k=1}^{n} \lambda_k\right) \end{cases}$$

If we define $\lambda := \Sigma_{j=1}^{M} \lambda_j$, then the compound Poisson process X_t has the same characteristic function, at all times t, as our process Z_t for the default counting function. The finite dimensional distributions of the two processes coincide as well, so Z_t and X_t are substantially the same process. This is easily checked by writing the finite dimensional distributions in terms of independent increments, while recalling that both Z_t and X_t have stationary independent increments.

Loss dynamics. The underlying GPL process Z_t is non-decreasing and takes arbitrarily large values, given large enough times. The portfolio cumulated loss and the rescaled number of defaults processes are non-decreasing, but limited to the interval [0, 1]. Thus, we consider the deterministic non-decreasing function $\Psi : \mathbb{N} \cup \{0\} \to [0,1]$ and we define the cumulated portfolio loss process $\bar{L}_t$ as:

$$L_t := \Psi_L(Z_t) := \min(Z_t, M') \quad \text{and} \quad \bar{L}_t := \frac{L_t}{M'} \tag{5}$$

where $1/M'$, with $M' \geq M > 0$, is the minimum jump for the loss process.

Remark 1. Notice that the loss is bounded within the interval [0, 1] by construction, but there is still the possibility that the loss jumps more than M times, where M is the number of names in the portfolio. If this is the case, we may check afterwards that the probability of such events is negligible. This happens in all our examples.

The marginal distribution of the cumulated portfolio loss process L_t can easily be calculated. We obtain:

$$L_t = \min(Z_t, M') = Z_t 1_{\{Z_t < M'\}} + M' 1_{\{Z_t \geq M'\}}$$

Since Z_t has a known distribution, the distribution of L_t can easily be derived as a by-product. The related density (defined on integer values since the law is discrete) is:

$$p_{L_t}(x) = p_{Z_t}(x) 1_{\{x<M'\}} + \mathbb{Q}\{Z_t \geq M'\} 1_{\{x=M'\}}$$

The density of $\bar{L}_t$ follows directly. Also the intensity of L_t, that is, the density of the absolutely continuous compensator of L_t (see, for example, Giesecke & Goldberg, 2005), can be calculated directly and is given by:

$$h_L(t) = \sum_{j=1}^{n} \min\left(\alpha_j, \left(M' - Z_{t^-}\right)^+\right) \lambda_j(t) \tag{6}$$

(see Brigo, Pallavicini & Torresetti, 2006, for the details). The intensity h goes to zero when Z exceeds M', which corresponds to total loss, as expected. Further, if all the possible integer jump sizes between one and M' are allowed, that is, if $\alpha_j = j$ and $n = M'$, the intensity h_L jumps whenever the cumulated portfolio loss process L jumps. The intensity jumps downwards, and this would seem to go in the opposite direction with respect to self-excitedness, which is considered a desirable feature of loss models in general. However, self-exciting features are embedded in our model, and they are embedded in the possibility of having several defaults in small intervals, contrary to most approaches to loss modelling. Consider, for example, just two names: instead of having the loss of one name increase the likelihood of default (intensity) of a second name, we have both names defaulting together immediately. This embeds self-excitedness, although in an extreme way. Finally, to view the effect of single names on each other one needs a more sophisticated formulation, based on the common Poisson shocks framework, leading to the GPCL model, which is analysed in Brigo, Pallavicini & Torresetti (2007).

Model limits. The GPL model we have introduced can be viewed as a particularly simple parameterisation of the market implied loss distribution dynamics. A positive feature is that the loss changes only by positive jumps, as should happen in any sensible loss model. Furthermore, this choice allows us to achieve a good calibration to market data, as we see in the following section. However, we are not making explicit assumptions on two important issues. First, we have not addressed possible ways to make our model consistent with single-name dynamics. Second,

A. DJi-Traxx index and tranche quotes in basis points on May 13, 2005, along with the bid-ask spreads

	Att-Det	Maturities			
		Three-year	Five-year	Seven-year	10-year
Index		38(4)	54(1)	65(3)	77(2)
Tranche	0–3	2,060 (100)	4,262 (118)	5,421 (384)	6,489 (124)
	3–6	72 (10)	173 (68)	398 (40)	590 (20)
	6–9	28 (6)	57 (6)	141 (17)	188 (15)
	9–12	13 (2)	31 (5)	72 (20)	87 (15)
	12–22	3 (1)	21 (3)	42 (13)	60 (10)

Note: index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium

B. DJi-Traxx index and tranche quotes in basis points on October 11, 2005, along with the bid-ask spreads

	Att-Det	Maturities			
		Three-year	Five-year	Seven-year	10-year
Index		23 (2)	38 (1)	47 (1)	58 (1)
Tranche	0–3	762 (26)	137 (26)	4,862 (76)	5,862 (74)
	3–6	20 (10)	95 (1)	200 (3)	515 (10)
	6–9	7 (6)	28 (1)	43 (2)	100 (4)
	9–12		12 (2)	27 (4)	54 (5)
	12–22		7 (1)	13 (2)	23 (3)

Note: index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium

C. Calibration error ε_i in (8), calculated with respect to the bid-ask spread, for tranches quoted on May 13, 2005

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		0.0	–0.1	0.3	0.0
Tranche	0–3	0.0	0.1	0.2	–0.2
	3–6	0.0	0.0	–0.2	0.0
	6–9	0.0	0.0	–0.3	0.1
	9–12	–0.1	0.1	–0.1	0.4
	12–22	0.0	0.0	–0.2	–0.3

D. Cumulated intensities, integrated up to tranche maturities, of the GPL model with $M' = 200$

α	$\Lambda(T)$			
	3y	5y	7y	10y
1	1.955	3.726	4.464	7.694
3	0.000	0.062	0.305	0.305
8	0.016	0.033	0.011	0.011
12	0.004	0.013	0.026	0.026
19	0.006	0.006	0.017	0.017
72	0.000	0.009	0.026	0.049
185	0.000	0.002	0.002	0.008

Note: each row corresponds to a different Poisson component with jump amplitude α. Recovery = 30%

we have not explained how to choose a full-featured pool spread and recovery dynamics. A possibility would be to make the intensities in the Poisson processes driving Z stochastic and to consider more general transformations of Z to obtain the loss process.

Since in the present article we focus on the calibration of CDO tranches, which depend only on the loss marginal distribution, we avoid discussing such problems here and we direct readers to Brigo, Pallavicini & Torresetti (2006) for an extensive analysis of candidate spread and recovery dynamics, and to Brigo, Pallavicini & Torresetti (2007) for further discussion, including consistency with single-name data. Significant progress and testing in loss modelling will be possible only when more liquid market quotes for tranche options and forward start tranches are available.

Model calibration

We work with the basic GPL model specification given by the driving GPL process Z in (3), which we use to model the pool loss through (5). In this basic formulation, each Poisson mode N_j has a deterministic piecewise-constant intensity $\lambda_j(t)$.

Given that we have modelled the pool loss $\bar{L}_t$ directly, we do not characterise the rescaled default counting process $\bar{C}_t$ completely, but give only its expectations, since these are the only information on default counting implicit in our market quotes (1) used for calibration (quotes (2) depend only on the loss and not on default counting explicitly). We thus assume:

$$\mathbb{E}_0\left[\bar{C}_t\right] := \frac{1}{1-\mathcal{R}}\mathbb{E}_0\left[\bar{L}_t\right] \quad \text{with} \quad 0 \le \mathcal{R} < 1 - \mathbb{E}_0\left[\bar{L}_{T_b}\right] \qquad (7)$$

where the range of definition of the constant $\mathcal{R}$ is taken to ensure that at each time t the expected value of the rescaled number of defaults is greater, or equal to, the cumulated portfolio loss, and that both are smaller or equal to one. Notice that we are avoiding introducing an explicit dynamics for the recovery rate (see Brigo, Pallavicini & Torresetti, 2006 and 2007, for an initial discussion on recovery dynamics). Our $\mathcal{R}$ here can be interpreted as a sort of average recovery rate.

Detailed calibration procedure. The model parameters found by the calibration procedure are the amplitudes $\alpha_j \in \{m \in \mathbb{N} : m \le M'\}$ with $j = 1 \dots n$, and the cumulated intensities $\Lambda_j(T)$, which are real non-decreasing piecewise linear functions in the tranche maturity.

The optimal values for the amplitudes α are selected in the following way:

- Fix the minimum jump size to $1/M'$ by choosing the integer $M' \ge M > 0$.
- Find the best integer value for α_1 by calibrating the cumulated intensity Λ_1 for each value of α_1 in the range $[1, M']$, all other modes being set to zero.
- Add the amplitude α_2 and find its best integer value by calibrating the cumulated intensities Λ_1 and Λ_2, starting from the previous value for Λ_1 as a guess, for each value of α_2 in the range $[1, M']$.
- Repeat the previous step for α_i with $i = 3$ and so on, by calibrating the cumulated intensities $\Lambda_1, \dots, \Lambda_i$, starting from the previously found $\Lambda_1, \dots, \Lambda_{i-1}$ as an initial guess, until the calibration error is under a given threshold or until the intensity Λ_i can be considered negligible.
- Check afterwards that the probability of having more than M jumps is negligible and that the value of $\mathcal{R}$ is within the arbitrage-free range given in (7).

The objective function f to be minimised in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda) = \sum_i \varepsilon_i^2, \quad \varepsilon_i = \frac{x_i(\alpha, \Lambda) - x_i^{Mid}}{x_i^{Bid} - x_i^{Ask}} \qquad (8)$$

where the x_i, with i running over the market quote set, are the

E. Calibration error ε_i in (8), calculated with respect to the bid-ask spread, for tranches quoted on Oct 11, 2005

	Att-Det	Maturities			
		Three-year	Five-year	Seven-year	10-year
Index		0.0	0.0	0.1	0.1
Tranche	0–3	–0.1	0.1	–1.2	2.1
	3–6	–0.1	–0.1	0.3	–1.0
	6–9	0.0	–0.1	0.3	0.9
	9–12		0.4	–0.8	–0.8
	12–22		0.0	–0.0	0.0

Note: the three-year maturity quotes lack two tranches

F. Cumulated intensities, integrated up to tranche maturities, of the GPL model with $M' = 200$

α	$\Lambda(T)$			
	Three-year	Five-year	Seven-year	10-year
1	0.441	2.498	4.466	7.555
2	0.435	0.435	0.435	0.671
11	0.004	0.023	0.023	0.023
22	0.004	0.001	0.006	0.030
29	0.000	0.000	0.001	0.001
32	0.000	0.004	0.004	0.004
192	0.000	0.001	0.005	0.011

Note: each row corresponds to a different Poisson component with jump amplitude α. Recovery = 30%

G. GPL calibration error for different minimum loss sizes $1/M'$ with respect to the bid-ask spread for 10-year tranches on October 11, 2005 (recovery = 30%)

	Att-Det	50bp	10bp	2bp
Tranche	0–3	2.1	1.8	1.8
	3–6	–1.0	–1.0	–1.0
10yr maturity	6–9	0.9	0.9	0.9
	9–12	–0.8	–0.9	–0.8
	12–22	0.0	0.2	0.0

index values S_0 for DJ iTraxx index quotes, and either the index periodic premium rates $S_0^{A,B}$ or the upfront premium rates $U^{A,B}$ for the DJ iTraxx tranche quotes.

■ **Calibration results.** The GPL model is calibrated to the market quotes observed weekly from May 6, 2005–October 18, 2005. We take $\mathcal{R}$ = 30%, following Albanese, Chen & Dalessandro (2003), as the reference value for the recovery rate in the DJ iTraxx Europe market for spot and forward contracts. The quality of our calibration below is not altered if we select a value $\mathcal{R}$ = 40%, which resembles the recovery typically used in simplified quoting mechanisms in the market (see Brigo, Pallavicini & Torresetti, 2006 and 2007, for some examples). We start with M' = 200, corresponding to a minimum loss jump size of 50bp.

Consider, as a first example, the calibration date May 13, 2005, whose inputs are given in table A. We list in tables C and D the calibration result and the values of the calibrated parameters. The calibration errors are very low for all maturities. Notice that a calibration error smaller than one means that the difference between the market quote and the model price is smaller than the bid-ask spread. Consider, as a second example, the calibration date October 11, 2005, whose inputs are given in table B. We list in tables E and F the calibration results and the values of the calibrated parameters. The calibration errors show that the 10-year equity tranche is not correctly priced. We find such mis-pricing in many calibration examples, in particular after October 2005.

H. Values of the poisson's amplitudes α/M' for different values of the minimum loss jump $1/M'$ (%)

50bp	Poisson's amplitudes						
Date	1	2	3	4	5	6	7
May 6, 2005	0.50	1.50	4.00	6.00	9.50	39.50	92.50
Sep 2, 2005	0.50	1.00	4.00	5.50	12.50	39.00	100.00
Oct 11, 2005	0.50	1.00	5.50	11.00	14.50	16.00	96.00
10bp	Poisson's amplitudes						
Date	1	2	3	4	5	6	7
May 6, 2005	0.10	1.50	4.60	5.90	9.60	39.60	53.00
Aug 5, 2005	0.20	1.10	1.40	8.10	11.30	49.00	62.40
Oct 11, 2005	0.10	0.70	1.00	6.30	11.50	14.50	93.70
2bp	Poisson's amplitudes						
Date	1	2	3	4	5	6	7
May 6, 2005	0.02	1.50	5.26	9.64	17.58	39.64	99.78
Aug 12, 2005	0.38	1.06	1.14	7.38	12.24	41.34	99.80
Oct 3, 2005	0.02	0.98	1.16	7.52	9.74	43.34	65.16
Oct 11, 2005	0.16	0.68	1.00	6.30	10.98	14.46	94.90

Note: only the calibration dates between May 6, 2005 and October 18, 2005 where the α/M' values change are listed

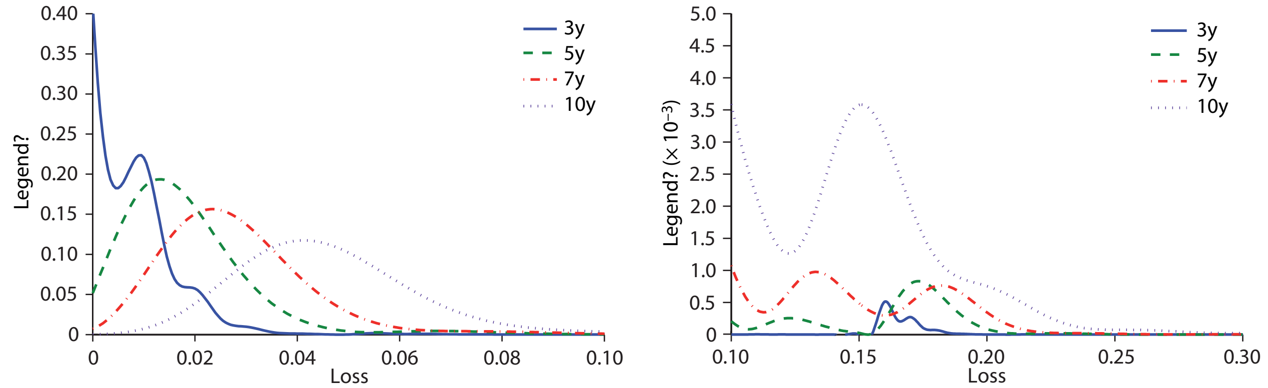
We then try as the minimum loss jump size $1/M'$, besides 50bp, the values 2bp and 10bp, corresponding, respectively, to M' equal to 5,000 and 1,000. Table G shows that the 10-year maturity tranches (the most difficult to calibrate in our experience) are stable through the three different loss sizes, suggesting that going below 50bp does not add much flexibility to the model. This is confirmed by further tests, and in particular the difference between the M' = 1,000 and the M' = 5,000 calibration is always small. Furthermore, the behaviour of the mean calibration error, that is, of the mean of the absolute values of the ε_is across time and quotes, for the three different choices of M is quite similar and within one.

We notice also that as the minimum jump size decreases ('granularity' increases), the loss distribution becomes noisier, due to the presence of small amplitudes. Furthermore, very small modes, appearing when the minimum jump size is as small as a few basis points, may violate the requirement that the loss process jumps fewer than M times (see Remark 1). We also tried calibrations with M less than 200, that is, with a minimum loss jump greater than 50bp. In this case, the calibration error grows quickly. Indeed, the minimum jump size, in this case, becomes greater than the typical portfolio loss given when one name defaults. Fifty basis points then seems to be a reasonable reference value.

The values of the Poisson amplitudes are quite stable across the calibration dates. Indeed, in six months we observe at most four changes in their values, as shown in table H.

The loss distribution implied by the GPL model is multi-modal and the probability mass moves towards larger loss values as the maturity increases. These features are shared by different approaches. For instance, static models, such as the implied default rate distribution in Torresetti, Brigo & Pallavicini (2006a) or the implied expected tranched-loss surface of Walker (2006) or Torresetti, Brigo & Pallavicini (2006b), predict multi-modal loss distributions. The

1 Loss distribution evolution of the GPL model



Note: minimum jump size of 50bp at all the quoted maturities up to 10 years, drawn as a continuous line

evolution of the implied loss distribution is shown in figure 1.

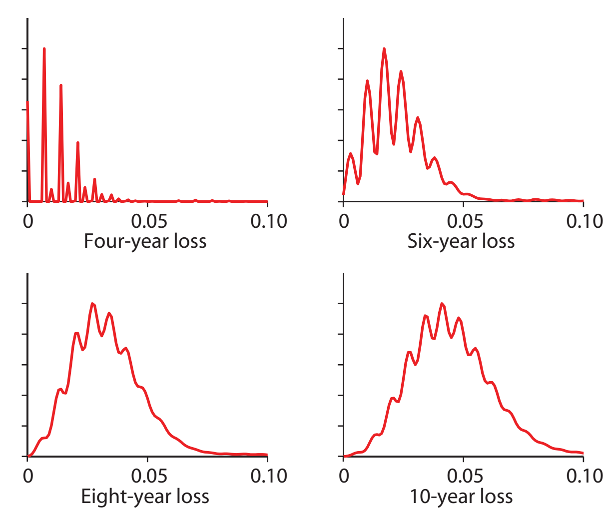
The dynamic credit correlation model of Albanese, Chen & Dalessandro (2005) shows implied loss distributions whose modes tend to group as the maturity increases, leading to a distribution approaching normality. The GPL model reproduces this behaviour, as shown in figure 2.

Conclusions and further research

We introduced the tractable and intuitive GPL dynamical model for the loss distribution of a pool. We presented detailed calibration results for the DJ iTraxx, which show good calibration outputs and relatively stable calibrated parameters over time. The default self-exciting feature of the model is embedded in allowing for more than one default in little time intervals.

Further research is needed for the GPCL generalisation, achieving consistency with single names, as we have begun in Brigo, Pallavicini & Torresetti (2007). Also, when liquid tranche options and forward-starting tranche quotes become available, testing of the tractable stochastic dynamical spread and recovery extensions hinted at in Brigo, Pallavicini & Torresetti (2006, 2007) should be considered. ■

2 Probability density of the cumulate portfolio loss process



Note: minimum loss jump size of 10bp for four-year, six-year, eight-year and 10-year maturities drawn as a continuous line on calibration date October 11, 2005

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