Back to the future

Current developments in exotic interest rate products push the demand for more sophisticated interest rate models. Here, Jesper Andreasen presents a new class of stochastic volatility multifactor yield curve models enabling quick calibration and efficient Monte Carlo simulation.
on a (discrete) basis of exponential functions. The function $k \mapsto \epsilon_k(t)$ can thus be viewed as the inverse Laplace transform of the forward rate volatility structure in the tenor dimension: $t \mapsto \sigma(t, t + \tau)$.

**Stochastic volatility processes**

The most popular stochastic volatility model for caps and swaptions appears to be the SABR model by Hagan et al (2002) where the volatility is specified as a geometric Brownian motion that has some correlation with the underlying forward swap rate. This model is quite difficult to work with in the context of full yield curve models, for a number of reasons.

First, the SABR model does not incorporate mean-reversion in volatility, which means that when the model is fitted to observed cap and swaption prices the implied volatility of volatility parameter most often turns out to be decreasing with the expiry of the underlying option. This in turn implies that a full dynamic version of the SABR model would have to exhibit even steeper decreasing forward volatility of volatility. Second, in many implementations of the SABR model the correlation between volatility and underlying rate are quite different for different expiries and tenors. Non-zero correlation is technically quite difficult to handle in a full yield curve model and potentially time-varying correlation is of course even more complicated. Third, as the SABR model has no closed-form for European-style option prices, it is typically implemented for European option pricing by expansion techniques whose accuracy deteriorates for longer expiries. This may have limited practical importance if the SABR model is only used for European-style option pricing, but our scope is to explain general path-dependent instruments, so we need our European-style option pricing to be consistent with the actual specified dynamics.

Instead we follow Andersen & Andreasen (2003) and use the following model as our basis for developing a full yield curve model with stochastic volatility:

$$dS(t) = \lambda \sqrt{\epsilon(t)} \left[ mS(t) + (1 - m)\epsilon(t) \right] dW^A(t)$$
$$dz(t) = \beta (1 - z(t)) dt + \epsilon \sqrt{\epsilon(t)} dW^B(t)$$

where $W^A$ is a Brownian motion under annuity measure, that is, the martingale measure with the annuity:

$$A(t) = \sum_{i=1}^{n} \delta_i P(t, t_i)$$

where $\delta_i = t_i - t_{i-1}$ is the day count fraction, $S(t) = \Pr(P(t_i, t_j) - P(t_j, t_i))A(t)$ is the forward par swap rate under consideration, and all the parameters $\lambda$, $m$, $\epsilon$, $\beta$ are constants. The swap rate is a martingale under the annuity measure.

In terms of the implied Black-Scholes smile, the level is controlled by $\lambda$. As correlation between the swap rate and the volatility is assumed to be zero, the slope of the smile is fully controlled by the $m$ parameter. The smile becomes increasingly negatively sloped as $m$ is decreased. Subnormal skews, corresponding to $m < 0$, are possible with the note of caution that $S$ is restricted from above by $\frac{2}{\lambda} \epsilon \sigma(0)$ when $m$ is negative. Increasing the volatility of local variance, $\epsilon$, increases the curvature of the smile. Increasing the speed of mean-reversion, $\beta$, increases the rate at which the curvature of the smile decays with expiry.

The model is essentially a ‘shifted’ Heston (1993) model, so it allows for an analytic solution based on numerical inversion of the Fourier transform. Lipton (2002) and Lewis (2000) give representations of the option price that avoid the numerical instability of the representation in the original Heston (1993) paper.

This model gives a good fit to observed cap and swaption prices with reasonably stable parameters across expiries and tenors. An example of the fitted $m$, $\epsilon$ parameters is given in table A. We note that the model's volatility of variance parameter, $\epsilon$, is related to lognormal volatility of volatility by the approximate relation:

lognormal volatility of volatility $\approx \epsilon / 2$

**A. Skew and smile parameters fitted to euro cap and swaption prices**

<table>
<thead>
<tr>
<th></th>
<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>5y</th>
<th>10y</th>
<th>15y</th>
<th>20y</th>
<th>30y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m$</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Note: this table reports best fit $m$ and $\epsilon$ parameters to observed cap and swaption prices for $\beta = 0.05$. Expiries are in the rows and tenors are in the columns. The currency is the euro and the parameters were estimated from Totem consensus prices for the end of a particular month in 2004.

**1. This and last month’s models against Totem quotes**

This figure shows the deviations from Totem consensus quotes for euro swaptions in terms of implied Black volatility for two models. One that had its $m$, $\epsilon$ parameters fitted the month before the other had its parameters fitted this month. Expiries range from six months to 20 years, tenors from one year to 30 years, and strikes from 5% to 95% in Black-Scholes delta terms – all in all, 912 swaptions. All data is as of a particular end of month in 2004.

In our experience, the implied skew and smile parameters $m$ and $\epsilon$ are quite stable over time, so in practice only the volatility level parameter $\lambda$ needs to be updated on a regular basis, say daily or weekly. To illustrate this, figure 1 shows the deviations in terms of implied Black volatility from Totem consensus prices for the end of a particular month in 2004.

Note: this table reports best fit $m$ and $\epsilon$ parameters to observed cap and swaption prices for $\beta = 0.05$. Expiries are in the rows and tenors are in the columns. The currency is the euro and the parameters were estimated from Totem consensus prices for the end of a particular month in 2004.

**1. This and last month’s models against Totem quotes**

This figure shows the deviations from Totem consensus quotes for euro swaptions in terms of implied Black volatility for two models. One that had its $m$, $\epsilon$ parameters fitted the month before the other had its parameters fitted this month. Expiries range from six months to 20 years, tenors from one year to 30 years, and strikes from 5% to 95% in Black-Scholes delta terms – all in all, 912 swaptions. All data is as of a particular end of month in 2004.

In our experience, the implied skew and smile parameters $m$ and $\epsilon$ are quite stable over time, so in practice only the volatility level parameter $\lambda$ needs to be updated on a regular basis, say daily or weekly. To illustrate this, figure 1 shows the deviations in terms of implied Black volatility from Totem consensus prices for the end of a particular month in 2004.

Note: this table reports best fit $m$ and $\epsilon$ parameters to observed cap and swaption prices for $\beta = 0.05$. Expiries are in the rows and tenors are in the columns. The currency is the euro and the parameters were estimated from Totem consensus prices for the end of a particular month in 2004.

1 Totem provides independent mark-certification services for interbank options based on mid-market quotes from approximately 20 leading option dealers.
2. Time-series evolution of calibrated $m$, $\varepsilon$ parameters

This figure shows the evolution of the monthly calibrated 10-year by 10-year skew and smile parameters $m$, $\varepsilon$ over the period 2003–05. The currency is the euro.

$m$ and $\varepsilon$ for the 10-year × 10-year euro swaption over 2003–05. Though the implied skew parameter $m$ increases over the period, the implied volatility of variance $\varepsilon$ is more or less constant over the period and both time-series exhibit low volatility.

Model specification

Andersen & Andreasen (2002) suggest a Libor market model with stochastic volatility, which is extended by Piterbarg (2003) to allow for a time- and tenor-dependent local volatility skew parameter. The motivation for this is that if we consider implied parameters of the model (4), as in table A, we typically see that the skew parameter $m$ is fairly constant across expiries but it tends to decrease with tenor. On the other hand, the implied $\varepsilon$ parameter appears to be fairly constant across both expiry as well as tenor, at least for expiries over one year.

If we use continuously compounded rates rather than discrete rates as model primitives, the Piterbarg model can be formulated as:

$$df(t) = \sqrt{\varepsilon(t)} \left[ m(t, T)f(t) + \left(1 - m(t, T)\right)f(0, T) \right] \lambda(t) dt + \sqrt{\varepsilon(t)} \varepsilon(t) dZ(t)$$

$$dc(t) = \beta(1 - \varepsilon(t)) dt + \sqrt{\varepsilon(t)} \varepsilon(t) dZ(t)$$

$$\lambda(t) \in \mathbb{R}, \varepsilon(t) \in \mathbb{R}^+, \|\varepsilon(t)\| = 1$$

$$Z(t) \in \mathbb{R}, dZ(t) dW(t) = 0$$

where $m$, $\lambda$, and $p$ are deterministic functions of time and maturity, $\varepsilon$ is a deterministic function of time and $\beta$ is a constant. We note that $m = 0$ corresponds to a normal model whereas $m = 1$ corresponds to a log-normal model.

Fix $k$ tenors $\tau_1, \ldots, \tau_k$. For the corresponding forward rates we have:

$$dF(t) = \sqrt{\varepsilon(t)} \left[ I_{M(t)} I_{F(t)} + (1 - I_{M(t)}) I_{F(t)} \right] \lambda(t) R(t) dW(t) + O(dt)$$

$$F(t) = \left(f(t, \tau_1), \ldots, f(t, \tau_k)\right) \in \mathbb{R}^k$$

$$M(t) = \left(m(t, \tau_1), \ldots, m(t, \tau_k)\right) \in \mathbb{R}^k$$

$$I_{M(t)} = \text{Diag}(\lambda(t, \tau_1), \ldots, \lambda(t, \tau_k)) \in \mathbb{R}^{k \times k}$$

$$R(t) = \left(p(t, \tau_1), \ldots, p(t, \tau_k)\right) \in \mathbb{R}^{k \times k}$$

Here the matrix product $RR^t$ is the instantaneous correlation matrix for the $k$ forward rates.

Under the separable volatility specification in (3), we have:

$$dF(t) = \Gamma(t) \eta(t) dW(t) + O(dt)$$

$$\Gamma(t) = \begin{bmatrix} g(t, t + \tau_1) \\ \vdots \\ g(t, t + \tau_k) \end{bmatrix} \in \mathbb{R}^{k \times k}$$

Equating diffusion terms of (6) and (7) yields:

$$\Gamma(t) \eta(t) = \sqrt{\varepsilon(t)} \left[ I_{M(t)} I_{F(t)} + (1 - I_{M(t)}) I_{F(t)} \right] \lambda(t) R(t)$$

$$\eta(t) = \sqrt{\varepsilon(t)} \Gamma(t)^{-1} \left( I_{M(t)} I_{F(t)} + (1 - I_{M(t)}) I_{F(t)} \right) \lambda(t) R(t)$$

with:

$$dc(t) = \beta(1 - \varepsilon(t)) dt + \varepsilon(t) dZ(t), dZ(t) dW(t) = 0$$

The volatility specification (7) in combination with (3) defines our model.

In most cases we choose constant $\kappa_1, \ldots, \kappa_k$ as well as a constant correlation structure $RR^t$. The latter is typically estimated from the historical time-series data of the yield curve. In this case, the model parameters that need to be set by calibration to swaption and cap prices are:

- The forward rate volatility structure, $\lambda$ for all times $t$ and the tenors $\tau_1, \ldots, \tau_k$.
- The forward rate skew structure, $m$ for all times $t$ and the tenors $\tau_1, \ldots, \tau_k$.
- The forward volatility of volatility, $\varepsilon$ for all times $t$.

In terms of the implied Black-Scholes volatility smiles for swaptions and caplets, the first parameter controls the absolute level, the second the slope (skew) and the third the curvature (smile).

We see that the model, at least in principle, can exactly fit the volatility level and slope for all expiries along $k$ tenors, whereas the curvature can only be fitted exactly for one tenor. In practice, though, our calibration will most often be on a best fit basis.

For the one-factor case, $k = 1$, we do, however, often choose to go for an exact fit to a specific strip of swaptions or caplets. In this case we often specify the model a bit differently, namely:

$$\eta(t) = \sqrt{\varepsilon(t)} \left[ m(t) S(t) + (1 - m(t)) S(0) \right] \lambda(t)$$

where $\lambda, m$ are now scalar functions of time and $S$ is a par swap rate referring to different swap periods over the time horizon. For example, if we choose to fit the model to the strip of 1×29, 2×28, …, 29×1 swap smiles, we let $S$ be the 1×29 par swap rate for times between year zero and one, 2×28 par swap rate for times between year one and two, up to 29×1 par swap rate for times between year 28 and 29.

Swaption pricing

For efficient calibration of the model, closed-form pricing of caps and swaptions is essential. In this section, we describe an accurate (near) closed-form approximation.

Using Itô’s lemma and the fact that the swap rate $S$ is a martingale under the annuity measure, we get:

$$dS(t) = S_x(t) \eta(t) dW^{R^t}(t)$$

where we let subscripts denote partial derivatives, that is, $S_x = (\partial S/\partial X_1, \ldots, \partial S/\partial X_k)$. Given fixed mean-reversion coefficients $\kappa_1, \ldots, \kappa_k$ this derivative can be calculated in closed form by combining (7) with the bond price formula in (3).

Our approximation goes in two steps:

- A. Approximate the stochastic differential equation (9) by the model:
where all parameters are time-dependent.

Approximation B. Approximate the stochastic differential equation (10) by the time-homogeneous model:

\[ \begin{align*}
\frac{d\tilde{S}(t)}{\tilde{S}(t)} &= \sqrt{\tilde{\sigma}(t)} \tilde{m} S(t) + \left(1 - \tilde{m}(t)\right) S(0) \frac{dW^A(t)}{	ilde{S}(t)} \\
\frac{d\tilde{z}(t)}{\tilde{z}(t)} &= \beta \left(1 - \tilde{z}(t)\right) dt + \tilde{z}(t) dW^A(t)
\end{align*} \]

(11)

where all parameters are constant.

Approximation A essentially involves finding time-dependent parameters \( \tilde{\lambda}, \tilde{m} \) so that the diffusion in (9) is approximated by the diffusion in (10), that is:

\[ \tilde{z}(t) = \frac{1}{S(0)^{1/2}} \left[ S_X(t) \tilde{\sigma}(t) \Gamma(t)^{-1} I_{F(0)}, R \right]_{Y=0} \]

(12)

Equating levels in (12) at \( X(t) = Y(t) = 0 \) yields:

\[ \tilde{\lambda}(t)^2 = \frac{1}{S(0)^{1/2}} \left[ S_X(t) \tilde{\sigma}(t) \Gamma(t)^{-1} I_{F(0)}, R \right]_{Y=0} \]

(13)

Differentiating (12) with respect to \( X, Y \) at \( X(t) = Y(t) = 0 \) yields:

\[ \left( \tilde{z}(t)^2 S(0) S_X(t) \right)_{Y=0} = \left[ \frac{1}{\tilde{\sigma}(t)} \frac{d}{dX} \left[ S_X(t) \eta(t) \right] \right]_{X=0, Y=0} \]

(14)

for \( i = 1, \ldots, k \). Due to the form of \( \eta(t) \), the right-hand side of (14) is independent of \( z(t) \), so (14) forms \( k \) linear equations in \( \tilde{m} \). We solve these by regression:

\[ \tilde{m}(t) = \left[ \begin{array}{cc}
\sum_{j=1}^{k} S_X(t) \eta(t) \end{array} \right]_{X=0, Y=0} \]

(15)

All quantities in (13) and (15) can be calculated in closed form using the zero-coupon bond price formula in (3).

It should be noted that this approximation can be slightly refined by evaluating (15) and (15) along levels of \( X, Y \) corresponding to approximate expected levels of \( X, Y \) under the annuity measure of the swaption under consideration.

Approximation B involves finding constant parameters so that the model (11) produces option prices that are close to those of (10) with parameters given by (13) and (15). We use the methodology suggested by Piterbarg (2005a-b). The exact details are quite complicated and are omitted here for space considerations, but the main point is that the technique is both very quick and accurate. Computationally, the method relies on a numerical solution of one Ricatti ordinary differential equation per swaption pricing – all remaining calculations are done in closed form. Compared with a direct solution of (10) by numerical inversion of Fourier transform as suggested in Andersen & Andreasen (2002), this technique is much faster and only marginally less accurate.

Calibration

We start by fixing \( \kappa_1, \ldots, \kappa_n \), the correlation structure for the forward rates \( RR^0 \), and a set of tenors \( \tau_1, \ldots, \tau_n \) of the model. We further fix a time grid \( 0 = t_0 < t_1 < \ldots \) of expiries and a set of tenors \( \{\tau_j\} \) corresponding to the swaption smiles that we wish to calibrate the model to. We assume that we have fitted parameters \( \tilde{\lambda}_i, \tilde{m}_i, \tilde{r}_i \) of the model (4) for these expiries (h) and tenors (j) of the calibration swaptions, as in table A.

We let the model (3) and (7) be parameterised by:

\[ \lambda(h, j) = \lambda_{h,i}, \quad m(h, j) = m_{h,i}, \quad \varepsilon(h) = \varepsilon_h \]

for \( t_0 < t_1 < \ldots \). We use approximation A and B to give us constant parameters \( \lambda_{h,0}, m_{h,0}, \varepsilon_h \) for each swaption. We now calibrate the model by bootstrapping, that is, we solve the optimisation problems:

\[ \min_{\lambda_{h,0}, m_{h,0}, \varepsilon_h} \sum_{j=1}^{n} \left[ \lambda_{h,j} - \lambda_{h,j} \right]^2 + \gamma_m \sum_{j=1}^{n} \left[ m_{h,j} - m_{h,j} \right]^2 + \gamma\varepsilon \sum_{j=1}^{n} \left[ \varepsilon_j - \varepsilon_j \right]^2 \]

sequentially for \( h = 1, 2, \ldots \). Here \( \gamma_{m}, \gamma_{\varepsilon}, \gamma \) are weights for balancing the different objectives against each other. Most often we calibrate the model in a sequence where only one of the weights \( \gamma_m, \gamma_{\varepsilon}, \gamma \) is non-zero at the time.

As an example of this, consider simultaneous calibration of a four-factor model of the type specified in (3) and (7) to all the euro cap and swap implied volatility smiles of 19 expiries ranging from six months to 20 years and eight tenors ranging from six months to 30 years. The implied volatility smiles are parameterised by the parameters in table A. We set:

\[ \left( \begin{array}{cccc}
\kappa_1, \kappa_2, \kappa_3, \kappa_4 \end{array} \right) = (0.015, 0.15, 0.30, 1.20) \]

\[ \left( \tau_1, \tau_2, \tau_3, \tau_4 \right) = (6m, 2y, 10y, 30y) \]

and use a correlation matrix estimated for historical time-series data of forward rate curves.

The resulting model parameters are shown in table B. We see that the forward skew parameters, \( m_j \), are decreasing more sharply in tenor than the corresponding ‘term’ skew parameters shown in table A. This is consistent with the findings in Piterbarg (2003). There does not appear to be a clear trend over time in any of the calibrated parameters. However, there is more noise in the calibrated forward skew parameters than in the Piterbarg (2003) case. This is probably due to the fact that we make no attempts to smooth our calibrated parameters in the time dimension. The calibration takes about five seconds of computer time.
Cutting edge  

RISK SEPTEMBER 2005 • WWW.RISK.NET

The error of such a calibration can be split in two. First, there is the error from the fact that a four-factor model will not be able to exactly match the smiles of eight tenors. We show this error by pricing swaptions and caps from the four-factor model and the target when we price caps and swaptions under our approximations A and B. Expiries range from six months to 30 years, tenors range from six months to 30 years and strikes range from 5–95% Black-Scholes delta – all in all 1,024 caplets and swaptions.

What actually counts, however, is of course what the error is when the model is simulated. We call this “total calibration error” and the result of pricing all the calibration swaptions by simulation is shown in Figure 4. We see that the total calibration error is within +/-0.40% in Black-Scholes volatility terms in most of the range.

In summary, a four-factor version of the model can simultaneously fit market prices of caps and swaptions for all strikes (5–95% delta), expiries (six months to 20 years), and tenors (six months to 30 years), within a tolerance of 0.4% in implied Black volatility terms. Moreover, the calibration only takes about five seconds of computer time.

Monte Carlo simulation

Strictly speaking, stochastic differential equations of the type defined by (3) and (7) can in some cases exhibit explosive behaviour. To avoid this problem, we follow Heath, Jarrow & Morton (1992) and simply replace \( f(t, r + \tau_j) \) in (7) with:

\[
\tilde{f}(t, r + \tau_j) = \max \left( f(0, r + \tau_j) - c, \min \left( f(t, r + \tau_j), f(0, r + \tau_j) + c \right) \right)
\]  

(16)

where \( c \) is some constant.

Due to the fact that the natural domain for the stochastic volatility factor \( z \) is \( \{ z \geq 0 \} \), straightforward Euler discretisation of the stochastic differential equation for \( z \) is going to exhibit very poor convergence as we decrease the time steps \( \Delta t \rightarrow 0 \). Instead, we prefer to use the following (local) lognormal discretisation:

\[
f(t, r + \tau_j) = f(0, r + \tau_j) \exp \left( \sqrt{\frac{\sigma \tau_j}{2}} \tilde{w}_j + \frac{\tilde{w}_j^2}{2} \right)
\]  

where \( \tilde{w}_j \) is a standard normal random variable.

\( f(t, r + \tau_j) \) is some constant.

Note: this table reports the resulting parameters when calibrating a four-factor model to the euro swaption and cap data of table A:

<table>
<thead>
<tr>
<th>maturity</th>
<th>LMM</th>
<th>HJM</th>
</tr>
</thead>
<tbody>
<tr>
<td>5y</td>
<td>2.12</td>
<td>1.14</td>
</tr>
<tr>
<td>10y</td>
<td>7.20</td>
<td>2.22</td>
</tr>
<tr>
<td>15y</td>
<td>15.19</td>
<td>3.33</td>
</tr>
<tr>
<td>20y</td>
<td>26.21</td>
<td>4.46</td>
</tr>
<tr>
<td>25y</td>
<td>40.27</td>
<td>5.53</td>
</tr>
<tr>
<td>30y</td>
<td>55.13</td>
<td>6.56</td>
</tr>
</tbody>
</table>

Note: CPU times in seconds for simulation of 5y, ..., 30y vanilla interest rate swaps with monthly reset in a four-factor Libor market model and our four-factor separable volatility structure HJM model.
where we choose $\bar{z}$, $\nu$ so that the lognormal approximation matches the two first conditional moments of $z(t_{n+1})$ given $z(t_n)$, that is:

\[
\bar{z} = 1 + e^{-\nu \Delta t} \left( z(t) - 1 \right)
\]

\[
v^2 = \ln \left[ 1 + \bar{z}^{-2} \left( 1 - e^{-\nu \Delta t} \right) + \frac{\nu^2}{2} \left( z(t) - 1 \right) \left( e^{-\nu \Delta t} - e^{-2\nu \Delta t} \right) \right]
\]

We combine this with standard Euler discretisation of $X, Y$. With typical parameter values, accurate pricing can be obtained with monthly or quarterly time stepping.

The strength of the separable volatility structure relative to the general HJM or LMM specification is the speed in simulation of the model. To illustrate this, we perform simulation of vanilla swaps with monthly rate reset in two models: an LMM with four factors and our separable model also with four factors. The resulting computer times are reported in table C. We see that in the LMM the computational time increases roughly with the square of the simulation horizon whereas it is linear for the separable model. Table C and our experience indicate that one can obtain computational savings of up to a factor 10 for longer-dated structures with the separable model relative to the LMM.

Finite difference solution

For the one-factor model, $k = 1$, finite difference solution is an efficient alternative to Monte Carlo simulation. The associated pricing partial differential equation can be written as:

\[
0 = \frac{\partial V}{\partial t} + \left[ D_y + D_z + D_z \right] V
\]

\[
D_y = \frac{-y}{3} \left( -kx + \gamma \right) \frac{\partial}{\partial x} + \frac{1}{2} \eta^2 \frac{\partial^2}{\partial x^2}
\]

\[
D_z = -\frac{z}{3} \left( \eta^2 - 2kx \right) \frac{\partial}{\partial y} + \frac{1}{2} \frac{\partial^2}{\partial y^2}
\]

\[
D_z = -\frac{z}{3} \beta \left( 1 - z \right) \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial^2}{\partial z^2}
\]

We use an alternating direction implicit scheme (see Mitchell & Griffiths, 1980) that splits the solution over each time step into three steps:

\[
\left[ \frac{1}{\Delta t} - \frac{1}{2} D_y \right] V(t + \frac{2}{3} \Delta t) = \left[ \frac{1}{\Delta t} - \frac{1}{2} D_z + D_z \right] V(t + \Delta t)
\]

\[
\left[ \frac{1}{\Delta t} - \frac{1}{2} D_y \right] V(t + \frac{2}{3} \Delta t) = \frac{1}{\Delta t} V(t + \frac{2}{3} \Delta t) - \frac{1}{2} D_z V(t + \Delta t)
\]

\[
\left[ \frac{1}{\Delta t} - \frac{1}{2} D_y \right] V(t + \frac{1}{3} \Delta t) = \frac{1}{\Delta t} \left( V(t + \frac{1}{3} \Delta t) - \frac{1}{2} D_z V(t + \Delta t) \right)
\]

(17)

where $V(t)$ is to be interpreted as a three-dimensional tensor of values at time $t$.

We use the standard three-point discretisation for $D_y$ and $D_z$, but for $D_z$ we use a five-point discretisation for the first derivative. This gives higher accuracy in the $y$ dimension, $O(\Delta y^4)$, and enables us to get away with relatively few $y$ steps, say 10. The disadvantage of the five-point discretisation is that the workload increases at a rate higher than the $O(\Delta y^4)$ of a three-point scheme but we find that is worth it in this particular case.

Square-root processes such as (7b), with high volatility and low mean reversion and therefore high probability of hitting $z = 0$ can be tricky to solve numerically. Linear discretisation of the $z$ axis according to the standard deviation of $z$ at maturity leads to very few points in the interval $[0, 1]$ relative to the number of points between one and the upper bound of $z$. Attempting to solve this problem by transforming the state variable introduces infinite drift for the transformed variable at $z = 0$ and this is therefore not a recommendable route. Instead we choose to discretise $z$ according to $\Delta z = O(\Delta t^2)$. This means that we get lower asymptotic accuracy than $O(\Delta t^3)$ but this does not seem to be a problem in practice.

In summary, we have a scheme with the following properties:

- Uniform von Neuman stability.
- Accuracy of $O(\Delta t^2 + \Delta y^4 + \Delta z^2)$, $p < 2$.
- Workload of $O(\Delta t^3 \times \Delta y^4 \times \Delta z^3)$, $q > 1$.

In practice, a 30-year Bermuda swaption is accurately priced on a grid of dimension $50 \times 100 \times 10 \times 15 \times 15$ (x x y x z) steps and this takes about three seconds of computer time.

Conclusion

We have presented a class of stochastic volatility yield curve models with quick and accurate calibration and significantly quicker Monte Carlo simulation than general HJM or Libor market models. A one-factor version of the model can be implemented with a finite difference solution and can thus be used as an alternative to the standard one-factor models for day-to-day management of large portfolios of interest rate exotics.

Jesper Andreasen is a principal in the fixed income quantitative research group at Bank of America in London. Email: jesper.andreasen@bankofamerica.com